Chapter 1

Matrices

Success is not where you are in life, but the obstacles you have overcome.
– Booker T. Washington

1.1 Matrices and their properties

Def. Matrix : A set of mn numbers (real or imaginary) arranged in the form of a rectangular array of m rows and n columns is called an $m \times n$ matrix (to be read as 'm' by 'n' matrix).

An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

In compact form the above matrix is represented by $A = \left[a_{ij}\right]_{m \times n}$ or $A = [a_{ij}]$.

The numbers $a_{11}, a_{12}, \ldots$ etc. are known as the elements of the matrix A. The element a_{ij} belongs to i^{th} row and j^{th} column and is called the $(i, j)^{\text{th}}$ element of the matrix $A = [a_{ij}]$. Thus, in the element a_{ij} the first subscript i always denotes the number of row and the second subscript j, number of column in which the element occurs.

Note : Unless otherwise stated, by a matrix we shall mean a complex matrix.

Types of matrices :

Def. Row matrix : A matrix having only one row is called a row-matrix or a row-vector.

e.g., $A=\begin{bmatrix}1 & 2 & -1 & -2\end{bmatrix}$ is a row matrix of order 1×4.

Def. Column matrix : A matrix having only one column is called a column matrix or a column-vector.

e.g., $A=\begin{bmatrix}1\\2\\-1\end{bmatrix}$ and $B=\begin{bmatrix}3\\2\\5\\4\end{bmatrix}$ are column-matrices of order 3×1 and 4×1 respectively.

Def. Square matrix : A matrix in which the number of rows is equal to the number of columns, say n, is called a square matrix of order n.

A square matrix of order n is also called a n-rowed square matrix. The elements a_{ij} of a square matrix $A=\left[a_{ij}\right]_{n\times n}$ for which $i=j$, i.e., the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal elements and the line along which they lie is called the principal diagonal or leading diagonal of the matrix.

e.g., the matrix $\begin{bmatrix}2 & 1 & -1\\3 & -2 & 5\\1 & 5 & -3\end{bmatrix}$ is square matrix of order 3 in which the diagonal elements are 2, −2 and −3.

Def. Identity or unit matrix : A square matrix $A=[a_{ij}]_{n\times n}$ is called an identity or unit matrix if

(i) $a_{ij}=0$ for all $i\neq j$ and (ii) $a_{ii}=1$ for all i

In other words, a square matrix each of whose diagonal element is unity and each of whose non-diagonal elements is equal to zero is called an identity or unit matrix. The identity matrix of order n is denoted by I_n.

e.g., the matrices $I_2=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $I_3=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ are identity matrices of order 2 and 3 respectively.

Def. Null matrix : A matrix whose all elements are zero is called a null matrix or a zero matrix.

e.g., $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ are null matrices of order 2×2 and 2×3 respectively.

Def. Equality of matrices : Two matrices $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{r\times s}$ are equal if

(i) $m=r$, i.e., the number of rows in A equals the number of rows in B

(ii) $n=s$, i.e., the number of columns in A equals the number of columns in B

(iii) $a_{ij}=b_{ij}$ for $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$

If two matrices A and B are equal, we write $A = B$, otherwise we write $A \neq B$. The matrices $A=\begin{bmatrix}3 & 2 & 1\\ x & y & 5\\ 1 & -1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3 & 2 & 1\\ -1 & 0 & 5\\ 1 & -1 & z\end{bmatrix}$ are equal if $x=-1$, $y=0$ and $z=4$. Matrices $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ are not equal, because their orders are not same.

Def. Addition of matrices : Let A, B be two matrices, each of order $m\times n$. Then their sum $A+B$ is a matrix of order $m\times n$ and is obtained by adding the corresponding elements of A and B. Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order, their sum $A+B$ is defined to be the matrix of order $m\times n$ such that $(A+B)_{ij}=a_{ij}+b_{ij}$ for $i=1,2,...,m$ and $j=1,2,...,n$

Note : The sum of two matrices is defined only when they are of the same order.

Properties of matrix addition :

1. Commutativity : If A and B are two $m\times n$ matrices, then $A+B=B+A$. i.e., matrix addition is commutative.
2. Associativity : If A, B, C are three matrices of the same order, then $(A+B)+C=A+(B+C)$. i.e., matrix addition is associative.
3. Existence of identity : The null matrix is the identity element for matrix addition, i.e., $A+O=A=O+A$.
4. Existence of inverse : For every matrix $A=[a_{ij}]_{m\times n}$ there exists a matrix $[-a_{ij}]_{m\times n}$, denoted by $-A$, such that $A+(-A)=O=(-A)+A$
5. Cancellation laws : If A, B, C are matrices of the same order, then

$A+B=A+C \Rightarrow B=C$ [Left cancellation law]

$B+A=C+A \Rightarrow B=C$ [Right cancellation law]

Def. Multiplication of a matrix by a scalar (scalar multiplication) : Let $A=[a_{ij}]$ be an $m\times n$ matrix and k be any number called a scalar. Then the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA. Thus, $kA=[ka_{ij}]_{m\times n}$.

Properties of scalar multiplication : If $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ are two matrices and k, l are scalars, then

(i) $k(A+B)=kA+kB$ (ii) $(k+l)A=kA+lA$

(iii) $(kl)A = k(lA) = l(kA)$ (iv) $(-k)A = -(kA) = k(-A)$

(v) $1A = A$ (vi) $(-1)A = -A$

Def. Subtraction of matrices : For two matrices A and B of the same order, the subtraction of matrix B from matrix A is denoted by $A - B$ and is defined as $A - B = A + (-B)$.

Def. Mulitiplication of matrices : Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post-multiplier). Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

$$\Rightarrow \quad (AB)_{ij} = [a_{i1}\, a_{i2} \ldots a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = (i^{\text{th}} \text{ row of } A)\,(j^{\text{th}} \text{ column of } B)$$

$$i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, p.$$

Remark : For any two matrices A and B, the following six possibilities about the products AB and BA may hold :

(i) AB and BA both does not exist.

(ii) AB exists but BA does not exist.

(iii) BA exists but AB does not exist.

(iv) Both AB and BA exist but their orders are not same.

(v) Both AB and BA exist, their orders are same but matrices are not equal i.e., $AB \neq BA$.

(vi) Both AB and BA exist and $AB = BA$.

Def. Positive integral powers of a square matrix : For any square matrix, we define

(i) $A^1 = A$ and (ii) $A^{n+1} = A^n \cdot A$, where $n \in \mathbb{N}$.

It is evident from this definition that $A^2 = AA$, $A^3 = A^2A = (AA)A$. etc.

It can be easily seen that (i) $A^m \cdot A^n = A^{m+n}$ and (ii) $(A^m)^n = A^{mn}$ for all $m, n \in \mathbb{N}$.

Properties of matrix multiplication :

1. Matrix multiplication is not commutative in general.
2. Matrix multiplication is associative i.e., $(AB)C = A(BC)$, whenever both sides are defined.
3. Matrix multiplication is distributive over matrix addition i.e.,

   (i) $A(B + C) = AB + AC$

   (ii) $(A + B)C = AC + BC$ whenever both sides of equality are defined.

4. If A is an $m\times n$ matrix, then $I_m A = A = AI_n$

5. The product of two matrices can be the null matrix while neither of them is the null matrix. i.e., if $A=\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, then $AB=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ while neither A nor B is the null matrix.

6. If A is $m\times n$ matrix and O is a null matrix, then

   (i) $A_{m\times n}O_{n\times p} = O_{m\times p}$ (ii) $O_{p\times m}A_{m\times n} = O_{p\times n}$

   i.e., the product of the matrix with a null matrix is always a null matrix.

7. In the case of matrix multiplication if $AB = O$, then it does not necessarily imply that $BA = O$. Let $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$. Then, $AB = O$. But, $BA=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\neq O$. Thus, $AB = O$ while $BA \neq O$.

8. Let A is $m\times n$ and B is $n\times p$ complex matrices. If sum of elements of each row (column) of A is k_1 and sum of elements of each row (column) of B is k_2, then sum of elements of each row (column) of AB is k_1k_2.

   Proof : We prove the result for 2×2 matrices and then it can be generalized easily to $n\times n$ matrices. Let $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ such that $a_{11}+a_{12}=k_1$, $a_{21}+a_{22}=k_1$

   and $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$ such that $b_{11}+b_{12}=k_2$, $b_{21}+b_{22}=k_2$

   then $AB=\begin{bmatrix}a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22}\\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}$ and

   Sum of elements of first row of $AB = a_{11}b_{11}+a_{12}b_{21}+a_{11}b_{12}+a_{12}b_{22}$

   $$= a_{11}(b_{11}+b_{12})+a_{12}(b_{21}+b_{22})$$
   $$= a_{11}(k_2)+a_{12}(k_2)$$
   $$= k_2(a_{11}+a_{12}) = k_1k_2$$

   Similarly, it can be proved that the sum of elements of second row is also k_1k_2.

9. Let A be a $n\times n$ complex matrix such that sum of elements of each row is k, then sum of elements of each row of A^2 is k^2.

   Proof : Take $B = A$ and $k_2 = k_1 = k$ in above result.

10. Let A be a $n \times n$ complex matrix such that sum of elements of each row is k, then sum of elements of each row of A^m is k^m, where m is a positive integer.

11. If A and B are two $n \times n$ complex matrices such that sum of elements of each row of A is k_1 and sum of elements of each row of B is k_2, then sum of all elements of AB is nk_1k_2.

12. Let A be a $n \times n$ complex matrix such that sum of each row of A is k, then sum of all elements of A^m is $n \cdot k^m$.

13. If A and B are two square matrices of same order such that $AB = BA$ and n is a positive integer, then $(A+B)^n$ can be expanded by binomial theorem i.e.,

$$(A+B)^n = {}^nC_0 A^n B^0 + {}^nC_1 A^{n-1} B^1 + \ldots + {}^nC_{n-1} A^1 B^{n-1} + {}^nC_n A^0 B^n.$$

14. If A and B are two square matrices of same order such that $AB = BA = 0$ and n is a positive integer, then $(A+B)^n = A^n + B^n$.

Def. Transpose of a matrix : If A is any matrix of order $m \times n$, then a matrix obtained from A by inter changing its rows and columns is called the transpose of the matrix A and is denoted by A' or A^T which is of order $n \times m$.

The operation of interchanging rows with columns is called transposition. Symbolically if $A = [a_{ij}]_{m\times n}$ then $A' = [a_{ji}]_{n\times m}$ i.e., $(i,j)^{th}$ element of $A = (j,i)^{th}$ element of A'.

Properties of transpose : If A' and B' denote transpose of A and B respectively, then

(i) $(A')' = A$

(ii) $(A+B)' = A' + B'$, where A and B are conformable for addition.

(iii) $(kA)' = kA'$, where k is any number, real or complex.

(iv) $(AB)' = B'A'$, where A and B are conformable for multiplication.

(v) $(A^n)' = (A')^n$, A being a square matrix and n is a positive integer.

Def. Conjugate of a matrix : If A is any matrix of order $m \times n$ whose elements are complex numbers, then a matrix obtained from A by replacing each of its elements by their corresponding complex conjugate is called the conjugate of A and is denoted by $\overline{A}$ where $\overline{A}$ is also of same order $m \times n$. Symbolically, if $A = [a_{ij}]_{m\times n}$ then $\overline{A} = [\overline{a}_{ij}]_{m\times n}$ where $\overline{a}_{ij}$ is complex conjugate of a_{ij}.

Note : If the elements of the matrix A are all real numbers then $\overline{A} = A$.

Properties of conjugate : If $\overline{A}$ and $\overline{B}$ denote the conjugate of A and B respectively, then

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$, where A and B are conformable for addition.

(iii) $\overline{(kA)} = \overline{k}\ \overline{A}$, where k is any complex number.

(iv) $\overline{(AB)} = \overline{A}\ \overline{B}$, where A and B are conformable for multiplication.

(v) $\overline{(A^n)} = (\overline{A})^n$, where A is a square matrix and n is a positive integer.

Def. Transposed conjugate of a matrix : If A is any matrix of order $m \times n$, then the transpose of the conjugate of A is called transposed conjugate of A and is denoted by A^θ or A^*. Symbolically if $A = [a_{ij}]_{m\times n}$, then $\overline{A} = [\overline{a}_{ij}]_{m\times n}$ and $A^* = (\overline{A})' = [\overline{a}_{ji}]_{n\times m}$.

Properties of transposed conjugate : If A^* and B^* denote the transposed conjugate of A and B respectively, then

(i) $(A^*)^* = A$

(ii) $(A+B)^* = A^* + B^*$, where A and B are conformable for addition.

(iii) $(kA)^* = \overline{k}\ A^*$, where k is any complex number.

(iv) $(AB)^* = B^* A^*$, where A and B are conformable for addition.

(v) $(A^n)^* = (A^*)^n$, where A is any square matrix and n is a +ve integer.

Note : If k is any real number then $(kA)^* = kA^*$ because in this case $\overline{k} = k$

Def. Determinants : Every square matrix can be associated to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det$A$ or $|A|$ or,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

Determinant of a square matrix of order 1 : If $A = [a_{11}]$ is a square matrix of order 1, then the determinant of A is defined as $|A| = a_{11}$ or, $|a_{11}| = a_{11}$

Determinant of a square matrix of order 2 : If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then the expression $a_{11}a_{22}-a_{12}a_{21}$ is defined as the determinant of A.

i.e., $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

Thus, the determinant of a square matrix of order 2 is equal to the product of the diagonal elements minus the product of off-diagonal elements.

Determinant of a square matrix of order 3 : If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then the expression $a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{32}a_{21}-a_{11}a_{23}a_{32}-a_{22}a_{13}a_{31}-a_{12}a_{21}a_{33}$ is defined as the determinant of A i.e.,

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow \quad |A|=(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}+(-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+(-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow \quad |A|=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Thus the determinant of a square matrix of order 3 is the sum of the product of elements a_{1j} in first row with $(-1)^{1+j}$ times the determinant of a 2×2 sub-matrix obtained by leaving the first row and column passing through the element.

Remark : For the determinant of a square matrix of order 3 can be arranged in various forms to obtain the expansion of $|A|$ along any of its rows or columns. Infact, to expand $|A|$ about a row or a column we multiply each element a_{ij} in i^{th} row with $(-1)^{i+j}$ times the determinant of the sub-matrix obtained by leaving the row and column passing through the element.

Determinant of a square matrix of order 4 or more : To calculate the determinant of a square matrix of order 4 or more, we expand $|A|$ about a row or a column by multiplying each element a_{ij} in i^{th} row with $(-1)^{i+j}$ times the determinant of the sub-matrix obtained by leaving the row and column passing through the element.

Def. Minor : Let $A=[a_{ij}]$ be a square matrix of order n. Then the minor M_{ij} of a_{ij} in A is the determinant of the square sub-matrix of order $(n-1)$ obtained by leaving i^{th} row and j^{th} column of A.

Def. Cofactor : Let $A=[a_{ij}]$ be a square matrix of order n. Then, the cofactor C_{ij} of a_{ij} in A is equal to $(-1)^{i+j}$ times the determinant of the sub-matrix of order $(n-1)$ obtained by leaving i^{th} row and j^{th} column of A. It follows from this definition that $C_{ij}=$ Cofactor of a_{ij} in $A=(-1)^{i+j}M_{ij}$, where M_{ij} is minor of a_{ij} in A. $C_{ij}=\begin{cases} M_{ij} & \text{if } i+j \text{ is even} \\ -M_{ij} & \text{if } i+j \text{ is odd}\end{cases}$

Properties of determinants :

Property 1 : Let $A=[a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with their cofactors is always equal to $|A|$ or, $\det(A)$

i.e., $\sum_{j=1}^{n} a_{ij}C_{ij}=|A|$ and $\sum_{i=1}^{n} a_{ij}C_{ij}=|A|$.

Property 2 : Let $A=[a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with the cofactors of the corresponding elements of some other row (column) is zero. i.e., $\sum_{j=1}^{n} a_{ij}C_{kj}=0$ and $\sum_{i=1}^{n} a_{ij}C_{ik}=0$.

Property 3 : The value of a determinant remains unchanged if its rows and columns are interchanged.

Property 4 : If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes by minus sign only.

Property 5 : If any two rows (columns) of a determinant are identical, then its value is zero.

Property 6 : If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant.

Property 7 : It follows from the above property that we can take out any common factor from any one row or any one column of a given determinant.

Property 8 : Let $A=[a_{ij}]$ be a square matrix of order n, then $|kA|=k^n|A|$, because k is common from each row of kA.

Property 9 : If each element of a row (column) of a determinant is expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

e.g., $\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Property 10 : If each element of a row (column) of a determinant is multiplied by the same constant and then added to the corresponding elements of some other row (column), then the value of the determinant remains same.

Property 11 : If each element of a row (column) of a determinant is zero, then its value is zero.

Property 12 : If A and B are square matrices of the same order, then $|AB| = |A||B|$.

Property 13 : If A and B are square matrices of the same order, then $|A+B| \neq |A|+|B|$, in general.

Property 14 : Vandermonde determinant : $\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = \prod_{\substack{i<j \\ i,j=1,2,\dots,n}} (a_j - a_i)$

Def. Singular and Non-singular Matrix : A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$.

Def. Adjoint of a square Matrix : Let $A = [a_{ij}]_{n\times n}$ be the square matrix of order n and A_{ij} be the corresponding cofactor of a_{ij} in $|A|$, then the matrix

$$B = [A_{ij}]' = [A_{ji}] = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

is called the adjoint of A and is denoted by adj. A. Thus the adjoint of a matrix is the transpose of the matrix formed by the co-factors of A. --

Properties of adjoint :

1. If A is any n-square matrix, then $A(\text{adj. } A) = (\text{adj. } A)\, A = |A|\, I_n$, where I_n is the unit matrix of order n.
2. If A is a non-singular matrix of order n, then $|\text{adj. } A| = |A|^{n-1}$.

   Proof : If A is a non-singular matrix of order n, then $A(\text{adj}.A) = |A| I_n$

   Taking determinant on both sides, we get $|A(\text{adj}.A)| = \big||A| I_n\big| \Rightarrow |A||\text{adj}A| = |A|^n \Rightarrow |\text{adj}A| = |A|^{n-1}$

3. If A is a non-singular matrix of order n, then $|\text{adj}.(\text{adj}.A)| = |A|^{(n-1)^2}$.

   Proof : If A is a non-singular matrix of order n, then $|\text{adj}A| = |A|^{n-1}$

   Replacing A by adj. A, we get $|\text{adj}.(\text{adj}.A)| = |\text{adj}.A|^{n-1} = \left(|A|^{n-1}\right)^{n-1} = |A|^{(n-1)^2}$

4. Generalization : If A is a non-singular matrix of order n, then $|\underbrace{\text{adj.adj....adj.}}_{k\text{ times}}(A)| = |A|^{(n-1)^k}$.

5. If A is a non-singular matrix of order n and k be any scalar then $adj(kA) = k^{n-1}adj(A)$

6. If A is a singular matrix of order n, then $|\text{adj.}A| = 0$.

7. Adjoint of a non-singular matrix is non-singular.

8. If A is a non-singular matrix of order n, then adj. (adj. A) $= |A|^{n-2}A$. In words, double adjoint of a matrix is the scalar multiple of that matrix.

Proof : If A is a non-singular matrix of order n, then $A(\text{adj.}A) = |A|I_n$

Replacing A by adj. A, we get

$$(\text{adj.}A)(\text{adj.}(\text{adj.}A)) = |\text{adj.}A|I_n = |A|^{n-1}I_n \qquad [\because |\text{adj.}A| = |A|^{n-1}]$$

Multiplying both sides by A, we get

$$A(\text{adj.}A)(\text{adj.}(\text{adj.}A)) = |A|^{n-1}A$$

$$\Rightarrow \quad |A|I_n(\text{adj.}(\text{adj.}A)) = |A|^{n-1}A \qquad [\because A(\text{adj.}A) = |A|I_n]$$

$$\Rightarrow \quad \text{adj. (adj. } A) = |A|^{n-2}A$$

9. If A and B are two square matrices of same order then $\text{adj}(A+B) \neq \text{adj}(A) + \text{adj}(B)$, in generally.

10. If A and B are two square matrices of same order then adj. (AB) = (adj. B) (adj. A)

11. If A is a square matrix then adj. A' = (adj. A)$'$, where A' is the transpose of matrix A.

Def. Inverse of a square matrix : Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I_n$, then the matrix A is said to be invertible and the matrix B is called inverse of the matrix A and is denoted by A^{-1}.

Note : (i) If the matrix B is inverse of A then A is the inverse of B.

(ii) For the products AB and BA both to be defined and be equal it is necessary that A and B are both square matrices of the same order. Thus non-square matrices cannot possess inverse.

Properties of inverse :

1. Inverse of a square matrix , if it exists, is unique.

2. A square matrix A is invertible if and only if A is non singular i.e. $|A| \neq 0$.

3. If A is a non singular matrix then $A^{-1} = \dfrac{\text{adj. } A}{|A|}$.

4. A singular matrix cannot have an inverse.

5. If A and B are two square matrices of order n such that A is non-singular and $AB=0$, then $B=0$.

Proof : Since A is non-singular i.e., $|A|\neq 0$, therefore A^{-1} exists.

Now, $AB=0 \Rightarrow A^{-1}AB=A^{-1}0$ (Pre multiplying both sides by A^{-1})

$\Rightarrow B=0$.

6. If A and B are two square matrices of order n such that $AB=I_n$, then $BA=I_n$.

Proof : We have, $B=BI_n=B(AB)=(BA)B$

$\Rightarrow B-(BA)B=0$

$\Rightarrow (I_n-BA)B=0$ ....(1)

Since $AB=I_n$, therefore $|AB|=|I_n|=1$

$\Rightarrow |A||B|=1\neq 0 \Rightarrow |B|\neq 0$

Using above result, (1) becomes $I_n-BA=0$

$\Rightarrow BA=I_n$

7. Let A be a non-singular matrix with integer entries, then the necessary and sufficient condition for A^{-1} to be with all integer entries is that $|A|=\pm 1$.

Proof : Suppose that $|A|=\pm 1$, then since $A^{-1}=\frac{1}{|A|}\text{adj}A=\pm\text{adj}A$

Since A has integer entries, therefore adj$A$ also has integer entries. Therefore, A^{-1} has all integer entries.

Conversely : Now suppose that A^{-1} has all integer entries.

Since $AA^{-1}=I$

$\Rightarrow |AA^{-1}|=|I|=1$

$\Rightarrow |A||A^{-1}|=1$

Since A and A^{-1} both have integer entries, therefore $|A|$ and $|A^{-1}|$ are integers and thus 1 can be factored as $1\cdot 1$ or $(-1)(-1)$.

Thus $|A^{-1}|=\pm 1$.

8. Reversal law for inverse : If A and B be non singular matrices then $(AB)^{-1}=B^{-1}A^{-1}$

9. Generalization of Reversal law for inverse : If $A_1, A_2, \ldots\ldots, A_n$ be non singular matrices of same order then $(A_1A_2\ldots\ldots A_n)^{-1}=A_n^{-1}A_{n-1}^{-1}\ldots\ldots A_2^{-1}A_1^{-1}$.

10. If A is non singular matrix then $(A^k)^{-1}=(A^{-1})^k$, where k is any positive integer.

11. If A is a non-singular matrix then $|A^{-1}| = \frac{1}{|A|}$.

12. If A is a non-singular matrix then $(A')^{-1} = (A^{-1})'$.

13. If A is a non-singular matrix then $(\text{adj}.A)^{-1} = \text{adj}.A^{-1}$

14. If A and B are invertible matrices such that $kA + lB$ is invertible then $kB^{-1} + lA^{-1}$ is also invertible.

Proof : $A^{-1}(kA + lB)B^{-1} = kB^{-1} + lA^{-1}$.

15. If A and B are invertible matrices such that $A + B$ is invertible then $A^{-1} + B^{-1}$ is also invertible.

Proof : Directly follows from above result.

16. If A and B are square matrices of same order such that $AB = BA = 0$ and $A + B$ is non-singular then $A^k + B^k$ is also non-singular where k is positive integer.

Proof : $A + B$ is non-singular $\Rightarrow$ $(A+B)^k$ is non-singular $\Rightarrow$ $A^k + B^k$ is non-singular.

17. If A and B are square matrices of same order such that $AB = BA = 0$ and $A + B$ is non-singular then $A^k - B^k$ is also non-singular where k is an odd positive integer.

Proof : We have, $(A+B)^2 = (A-B)^2$. But $A + B$ is non-singular

$\Rightarrow$ $(A+B)^2$ is non-singular $\Rightarrow$ $(A-B)^2$ is non-singular

$\Rightarrow$ $(A-B)$ is non-singular $\Rightarrow$ $(A-B)^k$ is non-singular

$\Rightarrow$ $A^k - B^k$ is non-singular [Since k is odd so $(A-B)^k = A^k - B^k$]

18. The reversal law for multiplication holds for adjoint, inverse, transpose and transposed conjugate but it does not hold for conjugate.

Def. Trace : The sum of the elements of a square matrix A lying along the principal diagonal is called the trace of A and is denoted by $\text{tr}(A)$.

Properties of trace : Let A and B be any $n \times n$ complex matrices and k is any complex number, then

1. $\text{tr}(kA) = k\,\text{tr}(A)$
2. $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$
3. $\text{tr}(AB) = \text{tr}(BA)$
4. $\text{tr}(AB) \neq \text{tr}(A)\text{tr}(B)$, in general. In other words, $\text{tr}(AB)$ may or may not be equal to $\text{tr}(A)\text{tr}(B)$.
5. $\text{tr}(A^*) \neq \text{tr}(A)$, in general.
6. $\text{tr}(A') = \text{tr}(A)$

7. $\text{tr}(A^*) = \overline{(\text{tr}(A))}$

8. Let $A = [a_{ij}]_{n\times n}$ be any complex matrix, then $\text{tr}(AA^*) = \text{tr}(A^*A) = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2$ = Sum of squares of modulus of each element of A.

Proof: We consider 2×2 matrix first and then it can be easily generalized. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

where $a_{ij} \in \mathbb{C}$, then $AA^* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} \\ \bar{a}_{12} & \bar{a}_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\bar{a}_{11} + a_{12}\bar{a}_{12} & a_{11}\bar{a}_{21} + a_{12}\bar{a}_{22} \\ a_{21}\bar{a}_{11} + a_{22}\bar{a}_{12} & a_{21}\bar{a}_{21} + a_{22}\bar{a}_{22} \end{pmatrix}$

$$= \begin{pmatrix} |a_{11}|^2 + |a_{12}|^2 & a_{11}\bar{a}_{21} + a_{12}\bar{a}_{22} \\ a_{21}\bar{a}_{11} + a_{22}\bar{a}_{12} & |a_{21}|^2 + |a_{22}|^2 \end{pmatrix}$$

$\Rightarrow \quad \text{tr}(AA^*) = |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2$

9. Let A be a $n\times n$ complex matrix, then trace of AA^* is a non-negative real number.

10. Let A be a $n\times n$ complex matrix, then $\text{tr}(AA^*) = 0$ iff A is a zero matrix.

11. Let A be a $n\times n$ real matrix, then $\text{tr}(AA') = \text{tr}(A'A) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2$ = Sum of squares of each element of A

12. Let A be a $n\times n$ real matrix, then trace of AA' (or $A'A$) is a non-negative real number.

13. Let A be a $n\times n$ real matrix, then $\text{tr}(AA') = 0$ iff A is a zero matrix.

14. Let A and B be $m\times n$ matrices, then $\text{tr}(A'B) = \text{tr}(BA') = \text{tr}(AB') = \text{tr}(B'A)$.

15. There does not exist two $n\times n$ matrices A and B such that $AB - BA = I_n$.

16. In trace cyclic rotation is allowed. e.g. $\text{tr}(BAB') = \text{tr}(AB'B)$.

17. $\text{tr}(ABCD) = \text{tr}(BCDA) = \text{tr}(CDAB) = \text{tr}(DABC)$.

Exercise 1.1

1. (i) Compute the adjoint of the matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ and verify that $(\text{adj}.A)A = |A|I$.

(ii) For the matrix $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, verify that $(\text{adj}.\ A)A = |A|I$.

2. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, verify that (i) $(\text{adj}.\ A)^{-1} = \text{adj}.\ A^{-1}$ (ii) $(A^{-1})^{-1} = A$

3. If $A=\begin{bmatrix}3 & 1\\ 4 & 0\end{bmatrix}$, $B=\begin{bmatrix}4 & 0\\ 2 & 5\end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$

4. If $A=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$ and $B=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$, $i^2=-1$ verify that (i) $(AB)'=B'A'$ (ii) $(AB)^{*}=B^{*}A^{*}$

5. If $A=\begin{bmatrix}2 & 3\\ 5 & -7\end{bmatrix}$ verify that $(A^2)'=(A')^2$

6. If $A=\begin{bmatrix}2+3i & i\\ 6+5i & 0\end{bmatrix}$, $B=\begin{bmatrix}i & 2i+1\\ 2-i & -i\end{bmatrix}$, verify that (i) $\overline{AB}=\overline{A}.\overline{B}$ (ii) $(AB)^{*}=B^{*}A^{*}$

7. If $A_\alpha=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$, then prove that

(i) $A_\alpha\cdot A_\beta=A_{\alpha+\beta}$ (ii) $(A_\alpha)^n=\begin{bmatrix}\cos n\alpha & \sin n\alpha\\ -\sin n\alpha & \cos n\alpha\end{bmatrix}$, for every positive integer n.

8. If a is a non-zero real or complex number. Use the principle of mathematical induction to prove that if $A=\begin{bmatrix}a & 1\\ 0 & a\end{bmatrix}$, then $A^n=\begin{bmatrix}a^n & na^{n-1}\\ 0 & a^n\end{bmatrix}$ for every positive integer n.

9. If $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, prove that $(aI+bA)^n=a^nI+na^{n-1}bA$, where I is a unit matrix of order 2 and $n>0$.

10. If $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$, then prove that $A^n=\begin{bmatrix}3^{n-1} & 3^{n-1} & 3^{n-1}\\ 3^{n-1} & 3^{n-1} & 3^{n-1}\\ 3^{n-1} & 3^{n-1} & 3^{n-1}\end{bmatrix}$ for every positive integer n.

11. Let A, B be two matrices such that they commute. Show that for any positive integer n

(i) $AB^n=B^nA$ (ii) $(AB)^n=A^nB^n$

12. Under what conditions is the matrix equation $A^2-B^2=(A-B)(A+B)$ is true ?

13. If A and B are square matrices of order n, then prove that A and B will commute iff $A-\lambda I$ and $B-\lambda I$ commute for every scalar λ.

14. Give an example of two matrices A and B such that

(i) $A\neq O,\ B\neq O,\ AB=O$ and $BA\neq O$ (ii) $A\neq O,\ B\neq O,\ AB=BA=O$

15. Suppose A is invertible. Show that if $AB=AC$, then $B=C$. Give an example of a nonzero matrix A such that $AB=AC$ but $B\neq C$.

16. Find 2×2 invertible matrices A and B such that $A+B\neq 0$ and $A+B$ is not invertible.

17. Let $A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$. (a) Find A^n. (b) Find B^n.

18. Let $A=\begin{bmatrix}5 & 2\\ 0 & k\end{bmatrix}$. Find all numbers k for which A is a root of the polynomial :

(a) $f(x)=x^2-7x+10$ (b) $g(x)=x^2-25$ (c) $h(x)=x^2-4$

19. Let $B=\begin{bmatrix}1 & 0\\ 26 & 27\end{bmatrix}$. Find a matrix A such that $A^3=B$.

20. Prove that for any A, there does not exist a matrix B such that $AB-BA=I$.

21. (i) Give an example of two matrices A and B such that $|A+B|\neq|A|+|B|$.

(ii) Give an example of two matrices A and B such that $|A+B|=|A|+|B|$.

Answers

12. $AB=BA$

14. (i) $A=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ (ii) $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

15. $A=\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$, $C=\begin{bmatrix}2 & 2\\ 0 & 0\end{bmatrix}$

16. $A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$; $B=\begin{bmatrix}4 & 3\\ 3 & 0\end{bmatrix}$

17. (a) $\begin{bmatrix}1 & 2n\\ 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & n & \frac{1}{2}n(n-1)\\ 0 & 1 & n\\ 0 & 0 & 1\end{bmatrix}$

18. (a) $k=2$, (b) $k=-5$ (c) none

19. $\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$

21. (i) $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ (ii) $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

1.2 Some special matrices

Def. Diagonal matrix : A matrix $A = [a_{ij}]_{n\times n}$ is called a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$

In other words, a square matrix is a diagonal matrix if its non-diagonal elements are zero.

Remark : An $n\times n$ diagonal matrix whose diagonal elements are $d_1, d_2, ..., d_n$ can be represented as diag. $(d_1, d_2, ..., d_n)$.

Properties :

1. The sum of two diagonal matrices of same order is again a diagonal matrix of that order. i.e., if

$$A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1' & & & \\ & d_2' & & \\ & & \ddots & \\ & & & d_n' \end{bmatrix}, \text{ then } A+B = \begin{bmatrix} d_1+d_1' & & & \\ & d_2+d_2' & & \\ & & \ddots & \\ & & & d_n+d_n' \end{bmatrix}$$

2. The product of two diagonal matrices of same order is again a diagonal matrix of that order. i.e., if

$$A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1' & & & \\ & d_2' & & \\ & & \ddots & \\ & & & d_n' \end{bmatrix}, \text{ then } AB = \begin{bmatrix} d_1d_1' & & & \\ & d_2d_2' & & \\ & & \ddots & \\ & & & d_nd_n' \end{bmatrix}$$

3. The scalar product of a diagonal matrix is again a diagonal matrix. i.e., if $A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$

and k is any complex number, then $kA = \begin{bmatrix} kd_1 & & & \\ & kd_2 & & \\ & & \ddots & \\ & & & kd_n \end{bmatrix}$.

4. If A is a diagonal matrix, then the positive integral powers of A is again a diagonal matrix. i.e., if

$$A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}, \text{ then } A^m = \begin{bmatrix} d_1^m & & & \\ & d_2^m & & \\ & & \ddots & \\ & & & d_n^m \end{bmatrix}, \text{ where } m \geq 0.$$

5. If A is a diagonal matrix and all the diagonal elements of A are non-zero, then A^{-1} is also a diagonal matrix. i.e., if $A=\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$, where $d_i \neq 0$ for $i=1,2,\ldots,n$, then

$$A^{-1}=\begin{bmatrix} 1/d_1 & & & \\ & 1/d_2 & & \\ & & \ddots & \\ & & & 1/d_n \end{bmatrix}.$$

6. If A is a diagonal matrix, then $A=A'$. In other words, transpose of a diagonal matrix is equal to the matrix itself.

7. Pre-multiplication by a diagonal matrix : If $D=\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ is a diagonal matrix of order n

and $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$ be any $n \times m$ matrix, then $DA=\begin{bmatrix} d_1a_{11} & d_1a_{12} & \cdots & d_1a_{1m} \\ d_2a_{21} & d_2a_{22} & \cdots & d_2a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ d_na_{n1} & d_na_{n2} & \cdots & d_na_{nm} \end{bmatrix}$,

i.e., i^{th} row of A is multiplied by d_i.

8. Post-multiplication by a diagonal matrix : If $D=\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ is a diagonal matrix and

$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ be any $m \times n$ matrix, then $AD=\begin{bmatrix} d_1a_{11} & d_2a_{12} & \cdots & d_na_{1n} \\ d_1a_{21} & d_2a_{22} & \cdots & d_na_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ d_1a_{m1} & d_2a_{m2} & \cdots & d_na_{mn} \end{bmatrix}$,

i.e., i^{th} column of A is multiplied by d_i.

9. The determinant of a diagonal matrix is equal to the product of its diagonal elements, i.e., if

$A=\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ is a diagonal matrix of order n, then $|A|=d_1d_2\ldots d_n$.

Def. Scalar matrix : A square matrix A is called a scalar matrix if A is a diagonal matrix and all diagonal elements of A are equal.

Mathematically, $A=\left[a_{ij}\right]_{n\times n}$ is a scalar matrix if

(i) $a_{ij}=0$ for all $i\neq j$ (ii) $a_{ii}=k$, where k is any number

Properties :

1. The sum of two scalar matrices of same order is again a scalar matrix of that order. i.e., if

$$A=\begin{bmatrix} k_1 & & & \\ & k_1 & & \\ & & \ddots & \\ & & & k_1 \end{bmatrix} \text{ and } B=\begin{bmatrix} k_2 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_2 \end{bmatrix}, \text{ then } A+B=\begin{bmatrix} k_1+k_2 & & & \\ & k_1+k_2 & & \\ & & \ddots & \\ & & & k_1+k_2 \end{bmatrix}.$$

2. The product of two scalar matrices of same order is again a scalar matrix of that order. i.e., if

$$A=\begin{bmatrix} k_1 & & & \\ & k_1 & & \\ & & \ddots & \\ & & & k_1 \end{bmatrix} \text{ and } B=\begin{bmatrix} k_2 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_2 \end{bmatrix}, \text{ then } AB=\begin{bmatrix} k_1k_2 & & & \\ & k_1k_2 & & \\ & & \ddots & \\ & & & k_1k_2 \end{bmatrix}.$$

3. The scalar product of a scalar matrix is again a scalar matrix. i.e., if $A=\begin{bmatrix} k_1 & & & \\ & k_1 & & \\ & & \ddots & \\ & & & k_1 \end{bmatrix}$ and k is any complex number, then $kA=\begin{bmatrix} kk_1 & & & \\ & kk_1 & & \\ & & \ddots & \\ & & & kk_1 \end{bmatrix}$.

4. If A is a scalar matrix, then the positive integral powers of A is also a scalar matrix. i.e., if

$$A=\begin{bmatrix} k & & & \\ & k & & \\ & & \ddots & \\ & & & k \end{bmatrix}, \text{ then } A^n=\begin{bmatrix} k^n & & & \\ & k^n & & \\ & & \ddots & \\ & & & k^n \end{bmatrix}, \text{ where } n\geq 0.$$

5. If $A(\neq O)$ is a scalar matrix, then A^{-1} is also a scalar matrix. i.e., if $A=\begin{bmatrix} k & & & \\ & k & & \\ & & \ddots & \\ & & & k \end{bmatrix}$, where

$k\neq 0$, then $A^{-1}=\begin{bmatrix} 1/k & & & \\ & 1/k & & \\ & & \ddots & \\ & & & 1/k \end{bmatrix}$.

6. If A is a scalar matrix, then $A=A'$. In other words, transpose of a scalar matrix is equal to the matrix itself.

7. The determinant of a scalar matrix is equal to the product of its diagonal elements. i.e., if

$A=\begin{bmatrix} k & & & \\ & k & & \\ & & \ddots & \\ & & & k \end{bmatrix}$ is a scalar matrix of order n, then $|A|=k^n$.

8. Scalar matrices are the only matrices which commute with every square matrix of its order.

Def. Upper triangular matrix : A matrix is an upper triangular matrix if all elements below the principal diagonal are zero. Mathematically, $A=\left[a_{ij}\right]_{n\times n}$ is an upper triangular matrix if $a_{ij}=0$ for all $i>j$.

Def. Lower triangular matrix : A matrix is called lower triangular matrix if all elements above the principal diagonal are zero. Mathematically, $A=\left[a_{ij}\right]_{n\times n}$ is a lower triangular matrix if $a_{ij}=0$ for all $i<j$.

Def. Triangular matrix : A matrix which is either upper triangular or lower triangular is called a triangular matrix.

Properties :

1. If A and B are two upper (lower) triangular matrices of same order, then $A+B$ is also an upper (lower) triangular matrix.
2. If A and B are two upper (lower) triangular matrices of same order, then AB is also an upper (lower) triangular matrix.
3. If A is an upper (lower) triangular matrix and k is any complex number, then kA is also an upper (lower) triangular matrix.
4. If A is an upper (lower) triangular matrix, then the positive integral powers of A is also an upper (lower) triangular matrix. Further, if $d_1,d_2,...,d_n$ are the diagonal elements of a triangular matrix, then $d_1^m,d_2^m,....,d_n^m$ are the diagonal elements of A^m.
5. If A is an upper (lower) triangular matrix and all the diagonal elements of A are non-zero, then A^{-1} is also an upper (lower) triangular matrix.

6. If A is an upper (lower) triangular matrix, then A' is a lower (upper) triangular matrix.
7. The determinant of upper (lower) triangular matrix is equal to the product of its diagonal elements.
8. The trace of upper (lower) triangular matrix is equal to the sum of its diagonal elements.
9. Every diagonal matrix is both upper and lower triangular.

Def. Super upper triangular matrix : A matrix is said to be super upper triangular matrix if all elements below and on the principal diagonal are zero. Mathematically, $A = [a_{ij}]_{n\times n}$ is a super upper triangular matrix if $a_{ij} = 0$ for all $i \geq j$.

Def. Super lower triangular matrix : A matrix is said to be super lower triangular matrix if all elements above and on the principal diagonal are zero. Mathematically, $A = [a_{ij}]_{n\times n}$ is a super lower triangular matrix if $a_{ij} = 0$ for all $i \leq j$.

Properties :

1. If A and B are two super upper (lower) triangular matrices of same order, then $A + B$ is also a super upper (lower) triangular matrix.
2. If A and B are two super upper (lower) triangular matrices of same order, then AB is also a super upper (lower) triangular matrix.
3. If A is a super upper (lower) triangular matrix and k is any complex number, then kA is also a super upper (lower) triangular matrix.
4. If A is a super upper (lower) triangular matrix, then the positive integral powers of A is also a super upper (lower) triangular matrix.
5. Inverse of a super upper (lower) triangular matrix does not exist.
6. If A is a super upper (lower) triangular matrix, then A' is a super lower (upper) triangular matrix.
7. The determinant of super upper (lower) triangular matrix is zero.
8. The trace of super upper (lower) triangular matrix is zero.

Def. Backward diagonal matrix : A matrix $A = [a_{ij}]_{n\times n}$ is called a backward diagonal matrix if $a_{ij} = 0$ for all i, j such that $i + j \neq n + 1$.

Properties :

1. The sum of two backward diagonal matrices of same order is again a backward diagonal matrix of that order.

i.e., if $A=\begin{bmatrix}0 & \cdots & 0 & d_1\\ 0 & \cdots & d_2 & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n & \cdots & 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & \cdots & 0 & d_1'\\ 0 & \cdots & d_2' & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n' & \cdots & 0 & 0\end{bmatrix}$, then $A+B=\begin{bmatrix}0 & \cdots & 0 & d_1+d_1'\\ 0 & \cdots & d_2+d_2' & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n+d_n' & \cdots & 0 & 0\end{bmatrix}$

2. The product of two backward diagonal matrices may or may not be a backward diagonal matrix. However, the product of two backward diagonal matrices of same order is always a diagonal matrix of that order.

i.e., if $A=\begin{bmatrix}0 & \cdots & 0 & d_1\\ 0 & \cdots & d_2 & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n & \cdots & 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & \cdots & 0 & d_1'\\ 0 & \cdots & d_2' & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n' & \cdots & 0 & 0\end{bmatrix}$, then $AB=\begin{bmatrix}d_1d_n' & & & \\ & d_2d_{n-1}' & & \\ & & \ddots & \\ & & & d_nd_1'\end{bmatrix}$.

3. The scalar product of a backward diagonal matrix is again a backward diagonal matrix. i.e., if

$A=\begin{bmatrix}0 & \cdots & 0 & d_1\\ 0 & \cdots & d_2 & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n & \cdots & 0 & 0\end{bmatrix}$ and k is any complex number, then $kA=\begin{bmatrix}0 & \cdots & 0 & kd_1\\ 0 & \cdots & kd_2 & 0\\ \cdots & \cdots & \cdots & \cdots\\ kd_n & \cdots & 0 & 0\end{bmatrix}$.

4. If A is a backward diagonal matrix, then the positive integral power of A may or may not be a backward diagonal matrix. However, the positive odd integral power of A is a backward diagonal matrix while the positive even integral power of A is a diagonal matrix. i.e., if $A=\begin{bmatrix}0 & \cdots & 0 & d_1\\ 0 & \cdots & d_2 & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n & \cdots & 0 & 0\end{bmatrix}$

then $A^{2m+1}=\begin{bmatrix}0 & \cdots & 0 & d_1^{m+1}d_n^m\\ 0 & \cdots & d_2^{m+1}d_{n-1}^m & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n^{m+1}d_1^m & \cdots & 0 & 0\end{bmatrix}$ and $A^{2m}=\begin{bmatrix}(d_1d_n)^m & & & \\ & (d_2d_{n-1})^m & & \\ & & \ddots & \\ & & & (d_nd_1)^m\end{bmatrix}$,

where $m>0$.

5. If A is a backward diagonal matrix and all the backward diagonal elements of A are non-zero, then A^{-1} is also a backward diagonal matrix. i.e., if $A=\begin{bmatrix}0 & \cdots & 0 & d_1\\ 0 & \cdots & d_2 & 0\\ \cdots & \cdots & \cdots & \cdots\\ d_n & \cdots & 0 & 0\end{bmatrix}$, where $d_i\neq 0$ for $i=1,2,\ldots,n$,

then $A^{-1}=\begin{bmatrix}0 & .. & 0 & 1/d_n\\ 0 & .. & 1/d_{n-1} & 0\\ .. & .. & .. & ..\\ 1/d_1 & .. & 0 & 0\end{bmatrix}$.

6. If A is a backward diagonal matrix, then A' is also a backward diagonal matrix.

7. If $A=\begin{bmatrix}0 & .. & 0 & d_1\\ 0 & .. & d_2 & 0\\ .. & .. & .. & ..\\ d_n & .. & 0 & 0\end{bmatrix}$ is a backward diagonal matrix of order n, then $\det(A)=(-1)^{\left[\frac{n}{2}\right]}d_1 \cdot d_2 \cdots d_n$

8. If $A=\begin{bmatrix}0 & .. & 0 & d_1\\ 0 & .. & d_2 & 0\\ .. & .. & .. & ..\\ d_n & .. & 0 & 0\end{bmatrix}$ is a backward diagonal matrix of order n,then $\operatorname{tr}(A)=\begin{cases}0 & \text{if } n \text{ is even}\\ d_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\end{cases}$

Def. Backward scalar matrix : A matrix $A=\left[a_{ij}\right]_{n\times n}$ is called a backward scalar matrix if

$$a_{ij}=\begin{cases}k & \text{if } i+j=n+1\\ 0 & \text{if } i+j\neq n+1\end{cases}$$

Properties :

1. The sum of two backward scalar matrices of same order is again a backward scalar matrix of that order.

i.e., if $A=\begin{bmatrix}0 & .. & 0 & k_1\\ 0 & .. & k_1 & 0\\ .. & .. & .. & ..\\ k_1 & .. & 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & .. & 0 & k_2\\ 0 & .. & k_2 & 0\\ .. & .. & .. & ..\\ k_2 & .. & 0 & 0\end{bmatrix}$, then $A+B=\begin{bmatrix}0 & .. & 0 & k_1+k_2\\ 0 & .. & k_1+k_2 & 0\\ .. & .. & .. & ..\\ k_1+k_2 & .. & 0 & 0\end{bmatrix}$.

2. The product of two backward scalar matrices may or may not be a backward scalar matrix. However, the product of two backward scalar matrices of same order is always a scalar matrix of that order.

i.e., if $A=\begin{bmatrix}0 & .. & 0 & k_1\\ 0 & .. & k_1 & 0\\ .. & .. & .. & ..\\ k_1 & .. & 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & .. & 0 & k_2\\ 0 & .. & k_2 & 0\\ .. & .. & .. & ..\\ k_2 & .. & 0 & 0\end{bmatrix}$, then $AB=\begin{bmatrix}k_1k_2 & & & \\ & k_1k_2 & & \\ & & \ddots & \\ & & & k_1k_2\end{bmatrix}$.

3. The scalar product of a backward scalar matrix is again a backward scalar matrix. i.e., if

$$A=\begin{bmatrix} 0 & .. & 0 & k_1 \\ 0 & .. & k_1 & 0 \\ .. & .. & .. & .. \\ k_1 & .. & 0 & 0 \end{bmatrix} \text{ and } k \text{ is any complex number, then } kA=\begin{bmatrix} 0 & .. & 0 & kk_1 \\ 0 & .. & kk_1 & 0 \\ .. & .. & .. & .. \\ kk_1 & .. & 0 & 0 \end{bmatrix}.$$

4. If A is a backward scalar matrix, then the positive integral power of A may or may not be a backward scalar matrix. However, the positive odd integral powers of A is a backward scalar matrix while the positive even integral power of A is scalar matrix. i.e., if $A=\begin{bmatrix} 0 & .. & 0 & k \\ 0 & .. & k & 0 \\ .. & .. & .. & .. \\ k & .. & 0 & 0 \end{bmatrix}$

$$\text{then } A^{2m+1}=\begin{bmatrix} 0 & .. & 0 & k^{2m+1} \\ 0 & .. & k^{2m+1} & 0 \\ .. & .. & .. & .. \\ k^{2m+1} & .. & 0 & 0 \end{bmatrix} \text{ and } A^{2m}=\begin{bmatrix} k^{2m} & & & \\ & k^{2m} & & \\ & & \ddots & \\ & & & k^{2m} \end{bmatrix}, \text{ where } m>0.$$

5. If $A(\neq O)$ is a backward scalar matrix, then A^{-1} is also a backward scalar matrix. i.e., if

$$A=\begin{bmatrix} 0 & .. & 0 & k \\ 0 & .. & k & 0 \\ .. & .. & .. & .. \\ k & .. & 0 & 0 \end{bmatrix}, \text{ where } k\neq 0, \text{ then } A^{-1}=\begin{bmatrix} 0 & .. & 0 & 1/k \\ 0 & .. & 1/k & 0 \\ .. & .. & .. & .. \\ 1/k & .. & 0 & 0 \end{bmatrix}$$

6. If A is a backward scalar matrix, then A' is also a backward scalar matrix.

7. If $A=\begin{bmatrix} 0 & .. & 0 & k \\ 0 & .. & k & 0 \\ .. & .. & .. & .. \\ k & .. & 0 & 0 \end{bmatrix}$ is a backward scalar matrix of order n, then $\det(A)=(-1)^{\left[\frac{n}{2}\right]}k^n$

8. If $A=\begin{bmatrix} 0 & .. & 0 & k \\ 0 & .. & k & 0 \\ .. & .. & .. & .. \\ k & .. & 0 & 0 \end{bmatrix}$ is a backward scalar matrix of order n, then $\operatorname{tr}(A)=\begin{cases} 0 & \text{if } n \text{ is even} \\ k & \text{if } n \text{ is odd} \end{cases}$

Exercise 1.2

1. Find all real triangular matrices A such that $A^2 = B$, where (a) $B=\begin{bmatrix} 4 & 21 \\ 0 & 25 \end{bmatrix}$ (b) $B=\begin{bmatrix} 1 & 4 \\ 0 & -9 \end{bmatrix}$.

2. Let $B=\begin{bmatrix} 1 & 8 & 5 \\ 0 & 9 & 5 \\ 0 & 0 & 4 \end{bmatrix}$. Find a triangular matrix A with positive diagonal entries such that $A^2 = B$.

3. Using only the elements 0 and 1, find the number of 3×3 matrices that are :
   (a) diagonal (b) upper triangular (c) non-singular and upper triangular. Generalize to $n\times n$ matrices.

4. Let $D_k = kI$, the scalar matrix belonging to the scalar k, show that :
   (a) $D_k A = kA$ (b) $BD_k = kB$ (c) $D_k + D_{k'} = D_{k+k'}$ (d) $D_k D_{k'} = D_{kk'}$

5. Suppose $AB=C$, where A and C are upper triangular.
   (a) Find 2×2 nonzero matrices A, B, C where B is not upper triangular.
   (b) Suppose A is also invertible. Show that B must also be upper triangular.

6. Give an example of two matrices A and B such that $AB = BA$ but neither A nor B is a scalar matrix.

7. How many $n\times n$ complex diagonal matrices A are there which satisfy $A^5 = A$.

8. How many $n\times n$ real diagonal matrices A are there which satisfy $A^5 = A$.

Answers

1. (a) $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} -2 & -3 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -7 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} -2 & 7 \\ 0 & 5 \end{bmatrix}$ (b) none 2. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

3. All entries below the diagonal must be 0 to be upper triangular, and all diagonal entries must be 1 to be non singular. (a) 8 ; 2^n (b) 2^6; $2^{n(n+1)/2}$ (c) 2^3; $2^{n(n-1)/2}$.

5. (a) $A=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C=\begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix}$ 6. $A=\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$

7. 5^n 8. 3^n

1.3 Symmetric, skew-symmetric, hermitian and skew-hermitian matrices

Def. Symmetric matrix : A square matrix A is said to be symmetric if $A' = A$. or

A square matrix $A = [a_{ij}]$ said to be symmetric if $a_{ij} = a_{ji}$ for all i and j.

Examples : $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & h & c \end{bmatrix}$, $\begin{bmatrix} 1 & 1-i & 2+i \\ 1-i & 2 & 3+2i \\ 2+i & 3+2i & 3 \end{bmatrix}$.

Def. Skew-symmetric matrix : A square matrix A is said to be skew-symmetric if $A' = -A$

or

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j.

Examples : $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & i & 2 \\ -i & 0 & 1-i \\ -2 & -1+i & 0 \end{bmatrix}$.

Properties :

1. The diagonal elements of a symmetric matrix are arbitrary.
2. The diagonal elements of a skew-symmetric matrix are all zero.
3. If A is a square matrix, then
   (i) $A + A'$ is symmetric (ii) $A - A'$ is skew-symmetric
4. If A is any square matrix, then AA' and $A'A$ are both symmetric.
5. If A is symmetric then kA is symmetric, where k is any number real or complex.
6. If A is skew-symmetric then kA is skew-symmetric, where k is any number real or complex.
7. If A and B are symmetric matrices of same order, then
   (i) $A+B,\ A-B,\ AB+BA$ are symmetric (ii) $AB - BA$ is skew-symmetric
   (iii) $BAB,\ ABA$ are symmetric.
8. If A and B are skew-symmetric matrices of same order, then
   (i) $A+B, A-B, AB-BA$ are skew-symmetric (ii) $AB + BA$ is symmetric
   (iii) $BAB,\ ABA$ are skew-symmetric.
9. If A and B are symmetric matrices of same order, then AB is symmetric iff $AB = BA$.
10. If A and B are skew-symmetric matrices of same order, then AB is skew-symmetric iff $AB = -BA$.
11. If A and B are square matrices of same order then $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.
12. If A is a symmetric matrix then A^n is also symmetric for all positive integers n.
13. If A is a skew-symmetric matrix then A^n is symmetric if n is a positive even integer and A^n is

skew–symmetric if n is positive odd integer.

14. If A is a non-singular and symmetric matrix then adj. A is also symmetric.
15. If A is a skew-symmetric matrix of order n, then adj. A is symmetric or skew-symmetric according as n is odd or even.
16. Determinant of a skew-symmetric matrix of odd order is always zero.
17. Determinant a skew-symmetric matrix of even order with integral entries is always a perfect square.
18. Zero matrix is the only matrix which is both symmetric and skew-symmetric.
19. Every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

Def. Hermitian matrix : A square matrix A is said to be hermitian if $A^* = A$

or

A square matrix $A = [a_{ij}]$ is said to be Hermitian if $a_{ij} = \overline{a_{ji}}$ for all i and j.

Examples : $\begin{bmatrix} 1 & 3-i \\ 3+i & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2+i & 1-2i \\ 2-i & 2 & 3i \\ 1+2i & -3i & 3 \end{bmatrix}$.

Def. Skew-hermitian matrix : A square matrix A is said to be skew-hermitian if $A^* = -A$

or

A square matrix $A = [a_{ij}]$ is said to be skew-hermitian if $a_{ij} = -\overline{a_{ji}}$ for all i and j.

Examples : $\begin{bmatrix} 0 & 3+5i \\ -3+5i & 2i \end{bmatrix}, \begin{bmatrix} 3i & 1+i & 2-3i \\ -1+i & 0 & 5+i \\ -2-3i & -5+i & -4i \end{bmatrix}$

Properties :

1. The diagonal element of a hermitian matrix are purely real.
2. The diagonal elements of a skew-hermitian matrix are either zero or purely imaginary.
3. If A is a square matrix, then
   (i) $A + A^*$ is hermitian (ii) $A - A^*$ is skew-hermitian
4. If A is any square matrix, then AA^* and A^*A are both hermitian
5. If A is hermitian then kA is hermitian, where k is any real number.
6. If A is skew-hermitian then kA is skew-hermitian, where k is any real number.
7. If A is hermitian then iA is skew-hermitian.
8. If A is skew-hermitian then iA is hermitian.

9. If A and B are hermitian matrices of same order, then

(i) $A+B,\ A-B,\ AB+BA$ are hermitian (ii) $AB-BA$ is skew-hermitian

(iii) $BAB,\ ABA$ are hermitian.

10. If A and B are skew-hermitian matrices of same order, then

(i) $A+B, A-B,\ AB-BA$ are skew-hermitian (ii) $AB+BA$ is hermitian

(iii) BAB, ABA are skew-hermitian

11. If A and B are hermitian matrices of same order, then AB is hermitian if and only if $AB=BA$.

12. If A and B are square matrices of same order then B^*AB is hermitian or skew-hermitian according as A is hermitian or skew-hermitian.

13. If A is a hermitian matrix then A^n is also hermitian for all positive integers n.

14. If A is a skew-hermitian matrix then A^n is hermitian or skew-hermitian according as n is a positive even integer or an odd integer.

15. Determinant of a Hermitian matrix is always real.

16. Determinant of a Skew-Hermitian matrix of

(i) Even order is always real (ii) Odd order is either zero or purely imaginary

17. Every square matrix is uniquely expressible as the sum of a hermitian matrix and a skew-hermitian matrix.

18. Every square matrix A can be uniquely expressed as $A=P+iQ$ where P and Q are hermitian matrices.

19. Every hermitian matrix A can be written as $A=B+iC$ where B is real symmetric and C is real skew- symmetric.

20. Every skew-hermitian matrix A can be written as $A=B+iC$ where B is real skew-symmetric and C is real symmetric.

21. A real matrix is hermitian iff it is symmetric.

22. A non-real symmetric matrix cannot be hermitian.

23. A non-real hermitian matrix cannot be symmetric.

24. A real matrix is skew-hermitian iff it is skew-symmetric.

25. A non-real skew-symmetric matrix cannot be skew-hermitian.

26. A non-real skew-hermitian matrix cannot be skew-symmetric.

Example 1 : Express the matrix $A=\begin{bmatrix} 4 & 5 & 3 \\ -2 & 7 & 8 \\ -4 & -6 & 5 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrix.

Solution : Let $A = \begin{bmatrix} 4 & 5 & 3 \\ -2 & 7 & 8 \\ -4 & -6 & 5 \end{bmatrix}$ then $A' = \begin{bmatrix} 4 & -2 & -4 \\ 5 & 7 & -6 \\ 3 & 8 & 5 \end{bmatrix}$

Now, $A + A' = \begin{bmatrix} 8 & 3 & -1 \\ 3 & 14 & 2 \\ -1 & 2 & 10 \end{bmatrix}$ and $A - A' = \begin{bmatrix} 0 & 7 & 7 \\ -7 & 0 & 14 \\ -7 & -14 & 0 \end{bmatrix}$

$\Rightarrow \quad \frac{1}{2}(A + A') = \begin{bmatrix} 4 & \frac{3}{2} & -\frac{1}{2} \\ \frac{3}{2} & 7 & 1 \\ -\frac{1}{2} & 1 & 5 \end{bmatrix}$ and $\frac{1}{2}(A - A') = \begin{bmatrix} 0 & \frac{7}{2} & \frac{7}{2} \\ -\frac{7}{2} & 0 & 7 \\ -\frac{7}{2} & -7 & 0 \end{bmatrix}$

Since $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ where $\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew-symmetric.

Therefore $\begin{bmatrix} 4 & 5 & 3 \\ -2 & 7 & 8 \\ -4 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 4 & \frac{3}{2} & -\frac{1}{2} \\ \frac{3}{2} & 7 & 1 \\ -\frac{1}{2} & 1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{7}{2} & \frac{7}{2} \\ -\frac{7}{2} & 0 & 7 \\ -\frac{7}{2} & -7 & 0 \end{bmatrix}$

Exercise 1.3

1. Find x, y, z such that A is symmetric, where (a) $A = \begin{bmatrix} 2 & x & 3 \\ 4 & 5 & y \\ z & 1 & 7 \end{bmatrix}$ (b) $A = \begin{bmatrix} 7 & -6 & 2x \\ y & z & -2 \\ x & -2 & 5 \end{bmatrix}$.

2. Suppose A and B are symmetric. Show that the following are also symmetric :
   (a) $A + B$ (b) kA, for any scalar k (c) A^2
   (d) A^n, for $n > 0$ (e) $f(A)$, for any polynomial $f(x)$.

3. Express the matrix $\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ as a sum of a symmetric and skew-symmetric matrices.

4. Express $A=\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & -2 & 0 \end{bmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

5. Find real numbers x, y, z such that A is hermitian, where $A=\begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix}$.

6. Suppose A is a complex matrix. Show that AA^* and A^*A are hermitian.

7. Let A be a square matrix. Show that (a) $A+A^*$ is hermitian, (b) $A-A^*$ is skew-hermitian, (c) $A=B+C$, where B is hermitian and C is skew hermitian.

8. Express the following matrices as a sum of hermitian and skew-hermitian matrices :

(i) $\begin{bmatrix} 2 & 1+i & 2+3i \\ 2-i & 1+2i & -i \\ 2-4i & 3-2i & 4+5i \end{bmatrix}$ (ii) $\begin{bmatrix} 7-4i & 2 & 3+5i \\ 3i & 6i & 1-4i \\ -3i & 2-i & 3 \end{bmatrix}$

9. Let A and B are two hermitian matrices and $A^2+B^2=O$ then show that $A=O$ and $B=O$.

Answers

1. (a) $x=4,\ y=1,\ z=3$, (b) $x=0,\ y=-6,\ z$ any real number.

3. $\begin{bmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}+\begin{bmatrix} 0 & \frac{5}{2} & -2 \\ -\frac{5}{2} & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 & 4 & \frac{7}{2} \\ -1 & 1 & 4 & -\frac{3}{2} \\ 4 & 4 & 7 & -\frac{1}{2} \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 \end{bmatrix}+\begin{bmatrix} 0 & 1 & 1 & -\frac{1}{2} \\ -1 & 0 & 2 & \frac{5}{2} \\ -1 & -2 & 0 & \frac{3}{2} \\ \frac{1}{2} & -\frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix}$

5. $x=3,\ y=0,\ z=3$

7. (c) Hint : Let $B=\frac{1}{2}\left(A+A^*\right)$ and $C=\frac{1}{2}\left(A-A^*\right)$

8. (i) $\frac{1}{2}\begin{bmatrix} 4 & 3+2i & 4+7i \\ 3-2i & 2 & 3+i \\ 4-7i & 3-i & 8 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 0 & -1 & -i \\ 1 & 4i & -3-3i \\ -i & 3-3i & 10i \end{bmatrix}$

(ii) $\frac{1}{2}\begin{bmatrix} 14 & 2-3i & 3+8i \\ 2+3i & 0 & 3-3i \\ 3-8i & 3+3i & 6 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} -8i & 2+3i & 3+2i \\ -2+3i & 12i & -1-5i \\ -3+2i & 1-5i & 0 \end{bmatrix}$

1.4 Orthogonal and Unitary matrices

Def. Orthogonal Matrix : A square matrix A is said to be orthogonal if $AA' = A'A = I$. It is clear from the definition that if the matrix A is orthogonal then $A^{-1} = A'$.

Examples : $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$

$$\begin{bmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & \cos\beta & \sin\beta & 0 \\ 0 & -\sin\beta & \cos\beta & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} i & \sqrt{2} \\ -\sqrt{2} & i \end{bmatrix}$$

Remark : In a matrix, if every row and column contains exactly one element from the set $\{1,-1\}$ and all other elements are zero, then the matrix is orthogonal.

Properties :

1. (i) The sum of two orthogonal matrices need not be orthogonal. i.e., if A and B are two orthogonal matrices of same order, then $A+B$ may or may not be orthogonal.

   (ii) If A and B are two orthogonal matrices of same order, then $A+B$ is orthogonal iff $AB' + BA' + I = O$

   Proof : Since A and B are orthogonal, therefore we have $AA' = I, \quad BB' = I$

   Suppose, $AB' + BA' + I = O$

   Now, $(A+B)(A+B)' = (A+B)(A'+B') = AA' + BA' + AB' + BB' = I + BA' + AB' + I = I$

   $\Rightarrow \quad A+B$ is orthogonal.

   Conversely : suppose $A+B$ is orthogonal. i.e., $(A+B)(A+B)' = I$

   $\Rightarrow \quad AA' + AB' + BA' + BB' = I$

   $\Rightarrow \quad I + AB' + BA' + I = I$

   $\Rightarrow \quad AB' + BA' + I = O$

2. The product of two orthogonal matrices of same order is again an orthogonal matrix of that order.
3. Generalization : If $A_1, A_2, \ldots, A_n$ be n orthogonal matrices of same order then their product $A_1A_2\ldots A_n$ is also orthogonal.
4. (i) The scalar multiple of orthogonal matrix need not be orthogonal. i.e., if A is an orthogonal matrix and k is any complex number, then kA may or may not be orthogonal.

(ii) If A is an orthogonal matrix and k is any complex number, then kA is orthogonal iff $k=\pm 1$.

Proof : Since A is an orthogonal matrix, therefore we have $AA'=I$

Now, $(kA)(kA)'=k^2AA'=k^2I=I$ iff $k^2=1$ i.e., $k=\pm 1$

5. If A is an orthogonal matrix, then any positive integral power of A is also an orthogonal matrix.
6. If A is an orthogonal matrix, then A^{-1} is also orthogonal.
7. If A is an orthogonal matrix, then A' is also orthogonal.
8. The determinant of an orthogonal matrix is ± 1.

OR

The absolute value of determinant of an orthogonal matrix is unity.

9. An orthogonal matrix is always non-singular.

Def. Proper and Improper orthogonal matrix : An orthogonal matrix is said to be proper or improper according as $|A|=1$ or $|A|=-1$

10. If A is an orthogonal matrix with $|A|=1$, then each element of A is equal to its cofactor.
11. If A is a orthogonal matrix with $|A|=-1$, then each element of A is equal to negative of its cofactor.

Def. Orthonormal set : A set of vectors $\{v_1, v_2, ..., v_n\}$ is said to be an orthonormal set if $<v_i, v_j>=0$ for $i\neq j$ and $<v_i, v_i>=1$ for all i. e.g. the set $\{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\}$ is an orthonormal set.

12. A matrix is orthogonal iff its rows (columns) form an orthonormal set.
13. The number of orthogonal matrix of order $n\times n$ with entries from the set $\{0,1,-1\}$ is $2^n n!$
14. $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^n=\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$, where n is positive integer.
15. $\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}^n=\begin{bmatrix}\cos n\theta & -\sin n\theta & 0\\ \sin n\theta & \cos n\theta & 0\\ 0 & 0 & 1\end{bmatrix}$, where n is positive integer.

Example 1 : Show that $\frac{1}{3}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}$ is orthogonal. Also find the inverse.

Solution : Let $A=\frac{1}{3}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}$ Then $A'=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$

and $AA'=\frac{1}{9}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}=\frac{1}{9}\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$

Similarly $A'A = I$. Hence A is orthogonal and $A^{-1} = A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

Def. Unitary Matrix : A square matrix A is said to be unitary if $AA^* = A^*A = I$. It is clear from the definition that if a matrix A is unitary then $A^{-1} = A^*$.

Examples : $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$, $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$, $\begin{bmatrix} \pm i & 0 & \dots & 0 \\ 0 & \pm i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \pm i \end{bmatrix}$, $\begin{bmatrix} 0 & \dots & 0 & \pm i \\ 0 & \dots & \pm i & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \pm i & \dots & 0 & 0 \end{bmatrix}$

Note : If each element of the matrix A is real, then $\bar{A} = A$

$\Rightarrow \quad A^* = (\bar{A})' = A' \quad \Rightarrow \quad AA^* = A^*A = I \quad \Rightarrow \quad AA' = A'A = I$

So, a unitary matrix whose elements are all real, is an orthogonal matrix.

Remark : In a matrix, if every row and column contains exactly one element from the set $\{e^{i\theta} : \theta \in \mathbb{R}\}$ (i.e., unit modulus) and all other elements are zero, then the matrix is unitary.

Properties :

1. (i) The sum of two unitary matrices need not be unitary. i.e., if A and B are two unitary matrices of same order, then $A+B$ may or may not be unitary.

   (ii) If A and B are two unitary matrices of same order, then $A+B$ is unitary iff

   $AB^* + BA^* + I = O$

   Proof : Since A and B are unitary, therefore we have $AA^* = I, \quad BB^* = I$

   Suppose, $AB^* + BA^* + I = O$

   Now, $(A+B)(A+B)^* = (A+B)(A^* + B^*)$

   $= AA^* + BA^* + AB^* + BB^*$

   $= I + BA^* + AB^* + I$

   $= I$

   $\Rightarrow \quad A+B$ is unitary.

   Conversely : suppose $A+B$ is unitary. i.e., $(A+B)(A+B)^* = I$

   $\Rightarrow \quad AA^* + AB^* + BA^* + BB^* = I$

   $\Rightarrow \quad I + AB^* + BA^* + I = I$

   $\Rightarrow \quad AB^* + BA^* + I = O$

2. The product of two unitary matrices of same order is again a unitary matrix of that order.
3. Generalization : If $A_1, A_2, \ldots, A_n$ be n unitary matrices of same order then their product $A_1 A_2 \ldots A_n$ is also unitary.
4. (i) The scalar multiple of unitary matrix need not be unitary. i.e., if A is a unitary matrix and k is any complex number, then kA may or may not be unitary.

   (ii) If A is a unitary matrix and k is any complex number, then kA is unitary iff $|k|^2 = 1$.

   Proof : Since A is a unitary matrix, therefore we have $AA^* = I$

   Now, $(kA)(kA)^* = k\bar{k}AA^* = |k|^2 I = I$ iff $|k|^2 = 1$.
5. If A is a unitary matrix, then any positive integral power of A is also a unitary matrix.
6. If A is a unitary matrix, then A^{-1} is also unitary.
7. If A is a unitary matrix, then A' is also unitary.
8. The conjugate of a unitary matrix is unitary.
9. The transposed conjugate of a unitary matrix is unitary.
10. The determinant of a unitary matrix has absolute value 1. In other words, if A is unitary matrix then $|A| = e^{i\theta}$, $\theta \in \mathbb{R}$ or $|A| = \dfrac{a+ib}{\sqrt{a^2+b^2}}$ where $a^2+b^2 \neq 0$.
11. A unitary matrix is always a non-singular matrix.
12. A real matrix is unitary iff it is orthogonal.
13. A non-real orthogonal matrix cannot be unitary.

    For example : $\begin{bmatrix} i & \sqrt{2} \\ -\sqrt{2} & i \end{bmatrix}$.
14. A non-real unitary matrix cannot be orthogonal.

Example 2 : Show that $\dfrac{1}{5}\begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix}$ is unitary and find A^{-1}.

Solution : Given that $A = \dfrac{1}{5}\begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix}$

Then $A' = \dfrac{1}{5}\begin{bmatrix} -1+2i & 2-4i \\ -4-2i & -2-i \end{bmatrix}$ and $\overline{(A')} = \dfrac{1}{5}\begin{bmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{bmatrix} = A^\theta$

Now $AA^\theta = \dfrac{1}{25}\begin{bmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{bmatrix}\begin{bmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{bmatrix} = \dfrac{1}{25}\begin{bmatrix} 1+4+16+4 & 0 \\ 0 & 4+16+4+1 \end{bmatrix}$

$$= \frac{1}{25}\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly $A^{\theta}A = I$. Hence A is unitary and hence $A^{-1} = \frac{1}{5}\begin{bmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{bmatrix}$ $[\because A^{-1} = A^{\theta}]$

Def. Normal Matrix : A square matrix A is said to be normal if $AA^* = A^*A$.

Properties :

1. (i) The sum of two normal matrices need not be a normal matrix. i.e., if A and B are two normal matrices of same order, then $A+B$ may or may not be a normal matrix.

   (ii) If A and B are two normal matrices of same order such that $AB^* = B^*A$ and $A^*B = BA^*$, then $A+B$ is also a normal matrix.

   Proof : Since A and B are two normal matrices, therefore we have $AA^* = A^*A$ and $BB^* = B^*B$.

   Now, $(A+B)(A+B)^* = (A+B)(A^*+B^*)$ $= AA^* + BA^* + AB^* + BB^*$

   $= A^*A + A^*B + B^*A + B^*B$ $= A^*(A+B) + B^*(A+B)$

   $= (A^*+B^*)(A+B)$ $= (A+B)^*(A+B)$

   $\Rightarrow$ $A+B$ is a normal matrix.

2. (i) The product of two normal matrices need not be normal. i.e., if A and B are two normal matrices of same order, then AB may or may not be a normal matrix.

   (ii) If A and B are two normal matrices of same order such that $AB^* = B^*A$ and $A^*B = BA^*$, then AB is also a normal matrix.

   Proof : Since A and B are two normal matrices, therefore we have $AA^* = A^*A$ and $BB^* = B^*B$.

   Now, $(AB)(AB)^* = (AB)(B^*A^*) = A(BB^*)A^* = AB^*BA^* = B^*(AA^*)B = B^*A^*AB = (AB)^*(AB)$

   $\Rightarrow$ AB is a normal matrix.

3. The scalar multiple of normal matrix is again a normal matrix.
4. If A is a normal matrix, then any positive integral power of A is also a normal matrix.
5. If A is a normal matrix, then A' is also a normal matrix.
6. Every real symmetric, real skew-symmetric, hermitian, skew-hermitian, real orthogonal and unitary matrix is normal.
7. A normal matrix need not be symmetric, skew-symmetric, hermitian, skew-hermitian, orthogonal or unitary.

Exercise 1.4

1. Check whether the following matrices are proper orthogonal or improper orthogonal :

(i) $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

2. Prove that a matrix $A=[a]$ of order 1 is orthogonal if and only if $a=\pm 1$.

3. Show that the matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal iff $A=\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ or $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ where $a^2+b^2=1$

4. If A is an orthogonal matrix and if $B=AP$, where P is non-singular, then prove that PB^{-1} is orthogonal.

5. If A is symmetric and P is orthogonal, then show that $P^{-1}AP$ is symmetric.

6. If A and B commute, then prove that $C'AC$ and $C'BC$ commute if C is orthogonal.

7. If A is skew-symmetric and $I+A$ is non-singular, then prove that $B=(I-A)(I+A)^{-1}$ is orthogonal.

8. Show that the following matrices are unitary and hence find their inverse :

(i) $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

(iii) $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1+i}{2} & \frac{i}{\sqrt{3}} & \frac{3+i}{2\sqrt{15}} \\ -\frac{1}{2} & \frac{1}{\sqrt{3}} & \frac{4+3i}{2\sqrt{15}} \\ \frac{1}{2} & -\frac{i}{\sqrt{3}} & \frac{5i}{2\sqrt{15}} \end{bmatrix}$

9. Show that the matrix $A=\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2=1$.

10. Show that if A is hermitain and P is unitary, then $P^{-1}AP$ is hermitian.

11. If A is unitary and hermitian matrix, then show that A is involutory $(A^2=I)$.

12. If A is unitary and $B=AP$ where P is non-singular then PB^{-1} is unitary.

13. If A is skew hermitian and $(A-I)$ is non-singular then $(A+I)(A-I)^{-1}$ is unitary.

14. Show that the following matrices are unitary :

(i) $\begin{bmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$ (ii) $\begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -i & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ i & 0 & 0 \end{bmatrix}$

15. What do you notice in above matrices.

16. Show that the following matrices are orthogonal and hence find their inverse.

(i) $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

(iii) $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

17. Find a 2×2 orthogonal matrix P whose first row is a multiple of (a) $(3,-4)$ (b) $(1, 2)$

18. Find a 3×3 orthogonal matrix P, whose first two rows are multiple of :

(a) (1,2,3) and (0,–3,2) (b) (1,3,1) and (1,0,–1)

19. Suppose A and B are orthogonal matrices. Show tht A^T, A^{-1}, AB are also orthogonal.

20. Which of the following matrices are normal ? $A=\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}, B=\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}, C=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

21. Determine which of the following matrices are unitary :

$A=\begin{bmatrix} \frac{i}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-i}{2} \end{bmatrix}$, $B=\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$, $C=\frac{1}{2}\begin{bmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{bmatrix}$

22. Suppose A and B are unitary. Show that A^*, A^{-1}, AB are unitary.

23. Determine which of the following matrices are normal $A=\begin{bmatrix} 3+4i & 1 \\ i & 2+3i \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 \\ 1-i & i \end{bmatrix}$.

Answers

1. (i) Improper orthogonal. (ii) Proper orthogonal (iii) The matrix is not orthogonal.

8. (i) $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ (iii) $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1-i}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ \frac{3-i}{2\sqrt{15}} & \frac{4-3i}{2\sqrt{15}} & -\frac{5i}{2\sqrt{15}} \end{bmatrix}$

16. (i) $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ (ii) $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

17. (a) $\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$, (b) $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$

18. (a) $\begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ 0 & \frac{-3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{-13}{\sqrt{182}} & \frac{2}{\sqrt{182}} & \frac{3}{\sqrt{182}} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} & \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{22}} & \frac{-2}{\sqrt{22}} & \frac{3}{\sqrt{22}} \end{bmatrix}$

20. A, C

21. A, B

23. A

1.5 Idempotent, Involutory and Nilpotent matrices

Def. Idempotent matrix : A square matrix A is said to be idempotent matrix if $A^2 = A$.

Examples : Zero matrix, identity matrix, $\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$, $\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

Remark : The matrix $A = [a_{ij}]_{n\times n}$ where $a_{ij} = \frac{1}{n}$ for all i, j and $n \geq 2$ is an idempotent matrix, i.e.,

$\begin{bmatrix} 1/n & 1/n & \dots & 1/n \\ 1/n & 1/n & \dots & 1/n \\ \vdots & \vdots & \vdots & \vdots \\ 1/n & 1/n & \dots & 1/n \end{bmatrix}$ is always an idempotent matrix and $I - A = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \dots & -\frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \dots & 1-\frac{1}{n} \end{bmatrix}$ is also an idempotent matrix.

Properties :

1. (i) The sum of two idempotent matrices need not be an idempotent matrix. i.e., if A and B are two idempotent matrices, then $A+B$ may or may not be an idempotent matrix.

   (ii) If A and B are idempotent matrices of same order, then $A+B$ is idempotent iff $AB = BA = O$.

   Proof : We have $A^2 = A, B^2 = B$

   Suppose $AB = BA = O$.

   Now, $(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + B^2 = A + B$

   $\Rightarrow$ $A+B$ is idempotent.

   Conversely : suppose $A+B$ is idempotent. i.e., $(A+B)^2 = A+B \Rightarrow A^2 + AB + BA + B^2 = A + B$

   $\Rightarrow A + AB + BA + B = A + B \Rightarrow AB + BA = O$ ......(1)

   Pre-multiplying (1) both sides by A, we get $AAB + ABA = O \Rightarrow AB + ABA = O$ ......(2)

   Post-multiplying (1) both sides by A, we get $ABA + BAA = O \Rightarrow ABA + BA = O$ ......(3)

   By (2) and (3), we get $AB = BA$ ......(4)

   By (1) and (4), we get $AB = BA = O$.

2. (i) The product of two idempotent matrices need not be an idempotent matrix. i.e., if A and B are two idempotent matrices, then AB may or may not be an idempotent matrix.

   (ii) If A and B are two idempotent matrices such that $AB = BA$, then the product AB is also idempotent.

   Proof : Since A and B are idempotent matrices, therefore we have $A^2 = A,\ B^2 = B$

   Now, $(AB)^2 = ABAB = AABB = A^2B^2 = AB$ $\quad (\because AB = BA)$

   $\Rightarrow$ AB is idempotent.

3. (i) The scalar product of an idempotent matrix need not be an idempotent matrix. i.e., if A is an idempotent matrix and k is any complex number, then kA may or may not be an idempotent matrix.

   (ii) If A is an idempotent matrix and k is any complex number, then kA is an idempotent matrix iff $k = 0,1$.

   Proof : Since A is an idempotent matrix, therefore we have $A^2 = A$

   Now, $(kA)^2 = k^2A^2 = k^2A = kA$ iff $k^2 = k$ i.e., $k = 0,1$.

4. If A is an idempotent matrix, then the positive integral powers of A is also an idempotent matrix.

5. If A is an idempotent matrix, then A' is also an idempotent matrix.

6. If A is an idempotent matrix, k is a complex number and n is a positive integer, then $(kI + A)^n$ can be expressed as the linear combination of I and A.

   Proof : As A is idempotent so $A^n = A$ for all $n \geq 1$. Further kI and A commute with each other so $(kI + A)^n$ can be expanded by Binomial Theorem, so

$$(kI + A)^n = k^nI^n + {}^nC_1(kI)^{n-1}A + {}^nC_2(kI)^{n-2}A^2 + \dots + {}^nC_{n-1}(kI)A^{n-1} + {}^nC_n \cdot A^n$$

$$= k^nI + {}^nC_1k^{n-1}A + {}^nC_2k^{n-2}A + \dots + {}^nC_n \cdot A$$

$$= k^nI + \left[{}^nC_1k^{n-1} + {}^nC_2k^{n-2} + \dots + {}^nC_n\right]A = k^nI + \left[(1+k)^n - k^n\right]A$$

7. If A is an idempotent matrix, then $I - A$ is also an idempotent matrix.

   Proof : Since A is an idempotent matrix, therefore $A^2 = A$.

   Now, $(I - A)^2 = I - 2A + A^2 = I - 2A + A = I - A$

8. The determinant of an idempotent matrix is always 0 or 1.

9. Converse is not true :- If determinant of a matrix is 0 or 1 then it may or may not be idempotent.

10. Contrapositive :- If determinant of a matrix is different from 0 and 1 then it cannot be idempotent.

11. The trace of a $n \times n$ idempotent matrix always belongs to the set $\{0,1,2,\dots,n\}$.

12. Converse is not true :- If trace of a $n \times n$ matrix belongs to the set $\{0,1,2,\dots,n\}$ then it may or may not be idempotent.

13. Contrapositive :- If trace of a $n \times n$ matrix does not belong to the set $\{0,1,2,\ldots,n\}$ then it cannot be idempotent.

14. Identity matrix is the only non-singular idempotent matrix. In other words, if A is idempotent and $A \neq I$ then $|A| = 0$.

15. If a diagonal matrix is idempotent then each diagonal element is either 0 or 1.

16. Number of $n \times n$ diagonal idempotent matrices is 2^n.

17. If A is idempotent then $\operatorname{adj}(A)$ is also idempotent.

Def. Involutory matrix : A square matrix A is said to be involutory matrix if $A^2 = I$.

Examples : I, $\begin{bmatrix} \pm1 & 0 & .. & 0 \\ 0 & \pm1 & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & \pm1 \end{bmatrix}, \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ .. & .. & .. & .. \\ 1 & \ldots & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \ldots & 0 & -1 \\ 0 & \ldots & -1 & 0 \\ .. & .. & .. & .. \\ -1 & \ldots & 0 & 0 \end{bmatrix}$.

If n is an even positive integer and $d_1, d_2, \ldots, d_{\frac{n}{2}}$ are non zero complex numbers then the matrix

$\begin{bmatrix} 0 & 0 & \ldots & \ldots & d_1 \\ 0 & 0 & \ldots & d_2 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \frac{1}{d_2} & 0 & \ldots & 0 \\ \frac{1}{d_1} & 0 & 0 & \ldots & 0 \end{bmatrix}$ is an involutory matrix, e.g., $\begin{bmatrix} 0 & 0 & 0 & 5+2i \\ 0 & 0 & 4 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{5+2i} & 0 & 0 & 0 \end{bmatrix}$.

If $n \geq 3$ is an odd positive integer and $d_1, d_2, \ldots, d_{\frac{n-1}{2}}$ are non-zero complex numbers then the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \iddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{\frac{n-1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{d_{\frac{n-1}{2}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \iddots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{d_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n}$$

is involutory. e.g. $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1+i & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{1+i} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 5}$ is involutory.

Properties :

1. (i) The sum of two involutory matrices need not be an involutory matrix. i.e., if A and B are two involutory matrices, then $A+B$ may or may not be an involutory matrix.

   (ii) If A and B are involutory matrices of same order, then $A+B$ is involutory iff $AB+BA+I=O$.

   Proof : We have $A^2 = I, \quad B^2 = I$

   Suppose, $AB+BA+I=O$

   Now, $(A+B)^2 = A^2 + AB + BA + B^2 = I + AB + BA + I = I + O = I$

   $\Rightarrow \quad A+B$ is involutory.

   Conversely : suppose $A+B$ is involutory. i.e., $(A+B)^2 = I \quad \Rightarrow \quad A^2 + AB + BA + B^2 = I$

   $\Rightarrow \quad I + AB + BA + I = I \quad \Rightarrow \quad AB + BA + I = O$

2. (i) The product of two involutory matrices need not be an involutory matrix. i.e., if A and B are two involutory matrices, then AB may or may not be an involutory matrix.

   For example : $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ are involutory but their product is not involutory.

   (ii) If A and B are two involutory matrices such that $AB = BA$, then the product AB is also involutory.

   Proof : Since A and B are involutory matrices, therefore we have $A^2 = I, B^2 = I$

   Now, $(AB)^2 = ABAB = AABB = A^2B^2 = I \qquad (\because AB = BA)$

   $\Rightarrow \quad AB$ is involutory.

3. (i) The scalar multiple of an involutory matrix need not be an involutory matrix. i.e., if A is an involutory matrix and k is any complex number, then kA may or may not be an involutory matrix.

(ii) If A is an involutory matrix and k is any complex number, then kA is an involutory matrix iff $k=\pm 1$.

Proof : Since A is an involutory matrix, therefore we have $A^2=I$

Now, $(kA)^2=k^2A^2=k^2I=I$ iff $k^2=1$ i.e., $k=\pm 1$.

4. If A is an involutory matrix then $A^{2n}=I$ and $A^{2n+1}=A$ where n is any positive integer.
5. If A is an involutory matrix, then A' is also an involutory matrix.
6. Let A be an involutory matrix. If k is any complex number and n be a positive integer then $(kI+A)^n$ can be expressed as a linear combination of I and A.

Proof : $(kI+A)^n={}^nC_0(kI)^n+{}^nC_1(kI)^{n-1}A+{}^nC_2(kI)^{n-2}A^2+{}^nC_3(kI)^{n-3}A^3+....$

$$={}^nC_0k^nI+{}^nC_1k^{n-1}A+{}^nC_2k^{n-2}I+{}^nC_3k^{n-3}A+....$$

$$=\left({}^nC_0k^n+{}^nC_2k^{n-2}+....\right)I+\left({}^nC_1k^{n-1}+{}^nC_3k^{n-3}+....\right)A$$

7. If any two rows (columns) of the identity matrix are interchanged the resulting matrix is an involutory matrix.
8. A square matrix A is involutory iff $\frac{1}{2}(A+I)$ is idempotent. This relation gives a bijection between involutory matrices and idempotent matrices.
9. The determinant of an involutory matrix is always 1 or –1.
10. Converse is not true : If determinant of a matrix is 1 or –1 then it may or may not be involutory.
11. Contrapositive : If determinant of a matrix is different from 1 and –1 then it cannot be involutory.
12. The trace of a $n\times n$ involutory matrix always belongs to the set $\{-n,-(n-1),....,-1,0,1,2,....,n\}$.
13. Converse is not true : If trace of a $n\times n$ matrix belongs to the set $\{-n,-(n-1),....,-1,0,1,2,....,n\}$ then it may or may not be involutory.
14. Contrapositive : If trace of a $n\times n$ matrix does not belong to the set $\{-n,-(n-1),....,-1,0,1,2,....,n\}$ then it cannot be involutory.
15. There does not exist a real matrix A of odd order such that $A^2=-I$. But there exists real matrices A of even order such that $A^2=-I$. For example, $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix}$.
16. If a diagonal matrix is involutory then each diagonal element is either 1 or –1.

17. Number of $n\times n$ diagonal involutory matrices is 2^n.

18. If A is involutory then $\text{adj}(A)$ is also involutory.

Def. Nilpotent matrix : A square matrix A is said to be nilpotent matrix if there exists a positive integer k such that $A^k = O$.

Examples : Zero matrix, $\begin{bmatrix} 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Remark : All super lower triangular and super upper triangular matrices are nilpotent.

Def. Index of nilpotency : Let A be a nilpotent matrix then the smallest positive integer k for which $A^k = O$ is called index of nilpotency of A.

Properties :

1. (i) The sum of two nilpotent matrices need not be nilpotent matrix. i.e., if A and B are two nilpotent matrices, then $A+B$ may or may not be nilpotent matrix. For example, consider the matrices $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and again consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

   (ii) If A and B are two nilpotent matrices of same order such that $AB = BA$, then $A+B$ is also nilpotent matrix.

   (iii) If A and B are two nilpotent matrices with index m and n such that $AB = BA$, then $A+B$ is nilpotent with index $\le m+n-1$.

2. (i) The product of two nilpotent matrices need not be nilpotent matrix. i.e., if A and B are two nilpotent matrices, then AB may or may not be nilpotent matrix. For example consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

   (ii) If A and B are two nilpotent matrices of same order such that $AB = BA$, then AB is also a nilpotent matrix.

   (iii) If A and B are two nilpotent matrices with index m and n such that $AB = BA$, then AB is nilpotent with index $\le \min\{m,n\}$.

3. If A and B are any square matrices of same order such that AB is nilpotent, then BA is also nilpotent. Further, (index of nilpotency of BA) $\le$ (index of nilpotency of AB) $+1$.

4. The scalar multiple of nilpotent matrix is again a nilpotent matrix. i.e., if A is nilpotent matrix and k is any complex number, then kA is also a nilpotent matrix.

5. Index of nilpotency of a nilpotent matrix cannot exceed its size. In other words if A is a $n\times n$ nilpotent matrix with index of nilpotency k then $k\le n$.

6. If A is a $n\times n$ nilpotent matrix then we must have $A^n = O$.

7. If A is a $n\times n$ matrix such that $A\ne O$, $A^2\ne O$, ..., $A^n\ne O$, then A cannot be nilpotent.

Def. Ceiling function : $\lceil x\rceil$ is the smallest integer greater than or equal to x, where x is any real number.

8. If A is a nilpotent matrix with index of nilpotent n and k is a positive integer then A^k is nilpotent matrix with index of nilpotent $\left\lceil \frac{n}{k}\right\rceil$, where $\lceil\cdot\rceil$ denotes the ceiling function.

9. Transpose of a nilpotent matrix is nilpotent.

10. Let A be a nilpotent matrix with index of nilpotency m. If k is a complex number and $n\ge m$ be a positive integer then $(kI+A)^n$ can be expressed as the linear combination of $I, A, A^2, ..., A^{m-1}$.

Proof :

$$(kI+A)^n = {}^nC_0(kI)^n + {}^nC_1(kI)^{n-1}A + {}^nC_2(kI)^{n-2}A^2 + ... + {}^nC_{m-1}(kI)^{n-m+1}A^{m-1} + {}^nC_m(kI)^{n-m}A^m + ... + {}^nC_nA^n$$

$$= k^nI + {}^nC_1k^{n-1}A + {}^nC_2k^{n-2}A^2 + ... + {}^nC_{m-1}k^{n-m+1}A^{m-1} \qquad [\because A^m = A^{m+1} = ... = A^n = 0]$$

11. The determinant of a nilpotent matrix is always 0 i.e., every nilpotent matrix is singular. However converse is not true, every singular matrix is not nilpotent. Contrapositive : If A is non singular matrix then A is not nilpotent.

12. The trace of a nilpotent matrix is always 0.

Exercise 1.5

1. Show that the following matrices are idempotent :

(i) $\begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & c & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & b & 1 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. What do you notice in above matrices.

3. If $AB = A$ and $BA = B$, then show that A and B are idempotent matrices.

4. Express the matrix $B=\begin{bmatrix}2&2&0\\0&3&0\\0&2&2\end{bmatrix}$ as the sum of a scalar matrix and an idempotent matrix and hence calculate B^5.

5. Show that the matrix $\begin{bmatrix}a&b\\ \frac{1-a^2}{b}&-a\end{bmatrix}$, $b\neq 0$ is always involutory.

6. Express the matrix $B=\begin{bmatrix}3&0&0&1\\0&3&1&0\\0&1&3&0\\1&0&0&3\end{bmatrix}$ as the sum of a scalar matrix and an involutory matrix and hence calculate B^3.

7. Show that the following matrices are nilpotent and also find their index of nilpotency.

(i) $A=\begin{bmatrix}0&1&1&0&0\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$ (ii) $B=\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$ (iii) $C=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$

(iv) $D=\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}$ (v) $E=\begin{bmatrix}0&0&2\\0&0&0\\0&0&0\end{bmatrix}$ (vi) $F=\begin{bmatrix}1&1&1\\-1&-1&-1\\0&0&0\end{bmatrix}$

(vii) $G=\begin{bmatrix}1&1&1\\-1&-1&-1\\1&1&0\end{bmatrix}$ (viii) $H=\begin{bmatrix}1&2&3\\-5&-10&-15\\3&6&9\end{bmatrix}$

8. Construct five different nilpotent matrices by taking hint from the matrix H in above question.

9. Express the matrix $B=\begin{bmatrix}2&2&2\\0&2&3\\0&0&2\end{bmatrix}$ as the sum of a scalar matrix and a nilpotent matrix and hence calculate B^{10}.

10. Give an example of two matrices A and B such that AB is nilpotent but neither A nor B is nilpotent.

11. Give an example of two idempotent matrices such that their sum is also idempotent.

Answers

4. $B=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}+\begin{bmatrix}0&2&0\\0&1&0\\0&2&0\end{bmatrix}$; $B^5=\begin{bmatrix}32&422&0\\0&243&0\\0&422&32\end{bmatrix}$

6. $B=\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{bmatrix}+\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$; $B^3=\begin{bmatrix}36&0&0&28\\0&36&28&0\\0&28&36&0\\28&0&0&36\end{bmatrix}$

7. (i) 4 (ii) 4 (iii) 2 (iv) 3 (v) 2 (vi) 2 (vii) 3 (viii) 2

9. $B=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}+\begin{bmatrix}0&2&2\\0&0&3\\0&0&0\end{bmatrix}$; $B^{10}=\begin{bmatrix}1024&10240&79360\\0&1024&15360\\0&0&1024\end{bmatrix}$

10. $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

Sr. No.	Matrix	Definition	A+B	AB	kA	$\lvert A\rvert$	tr(A)
1	Diagonal	$a_{ij}=0,\ i\neq j$	√	√	√	Product of diagonal elements	Sum of diagonal elements
2	Scalar	$a_{ij}=\begin{cases}0, & i\neq j\\ k, & i=j\end{cases}$	√	√	√	k^n	nk
3	Lower triangular	$a_{ij}=0,\ i<j$	√	√	√	Product of diagonal elements	Sum of diagonal elements
4	Upper triangular	$a_{ij}=0,\ i>j$	√	√	√	Product of diagonal elements	Sum of diagonal elements
5	Super lower triangular	$a_{ij}=0,\ i\leq j$	√	√	√	0	0

6	Super upper triangular	$a_{ij}=0,\ i\geq j$	√	√	√	0	0
7	Backward diagonal	$a_{ij}=\begin{cases}0 & , i+j\neq n+1\\ a_i & , i+j=n+1\end{cases}$	√	×	√	$(-1)^{\left[\frac{n}{2}\right]}(d_1d_2...d_n)$	$\begin{cases}0 & \text{if } n \text{ is even}\\ d_{\frac{n+1}{2}} & \text{if } n \text{ is odd}\end{cases}$
8	Backward scalar	$a_{ij}=\begin{cases}0 & , i+j\neq n+1\\ k & , i+j=n+1\end{cases}$	√	×	√	$(-1)^{\left[\frac{n}{2}\right]}k^n$	$\begin{cases}0 & \text{if } n \text{ is even}\\ k & \text{if } n \text{ is odd}\end{cases}$
9	Symmetric	$A'=A$	√	×	√	---	Sum of diagonal elements
10	Skew-symmetric	$A'=-A$	√	×	√	0 if n is odd	0
11	Hermitian	$A^*=A$	√	×	×	Real	Real
12	Skew-hermitian	$A^*=-A$	√	×	×	Real or purely imaginary	Either zero or purely imaginary
13	Orthogonal	$AA'=I$	×	√	×	± 1	Sum of diagonal elements
14	Unitary	$AA^*=I$	×	√	×	Unit modulus	Sum of diagonal elements
15	Normal	$AA^*=A^*A$	×	×	√	---	Sum of diagonal elements
16	Idempotent	$A^2=A$	×	×	×	0 or 1	$\{0,1,2,....,n\}$
17	Involutory	$A^2=I$	×	×	×	1 or -1	$\{-n,...,0,1,...,n\}$
18	Nilpotent	$A^k=0$	×	×	√	0	0

True-false exercise

1. The product of two $n\times n$ symmetric matrices may be non-symmetric.

2. The matrix $\begin{bmatrix}1&3&7\\4&-2&3\\2&4&1\end{bmatrix}$ is symmetric.

3. The sum of any two $n\times n$ symmetric matrices is symmetric.

4. For any square matrix A, the sum $A+A^T$ is symmetric.

5. For square matrices, if $AA=0$, then $A=0$.

6. Let A and B be $n\times n$ matrices. The (i,j) element in the product B^TA is given by the formula $\left(B^TA\right)_{ij}=\sum_{k=1}^{n}A_{kj}B_{ki}$.

7. For two $n\times n$ matrices, A and B, we have $\sum_{j=1}^{n}\left(B^T\right)_{sj}A_{ji}=\left(AB^T\right)_{si}$.

8. For a 4×4 matrix A, if $A^4=0$, then $A=0$.

9. If two 2×2 matrices A and B satisfy the equation $BA=0$, then either $A=0$ or $B=0$.

10. For $n\times n$ matrices, this cancellation law holds : If $AB=CA$ and $A\neq 0$, then $B=C$.

11. If three matrices A,B and C satisfy the equation $AB=CB$ and if $B\neq 0$, then it follows that $A=C$.

12. $\begin{bmatrix}1&2\\2&-1\\1&2\end{bmatrix}\begin{bmatrix}3&1&2&1\\3&-1&1&0\end{bmatrix}=\begin{bmatrix}10&-2&5&1\\3&3&3&2\\9&-1&4&1\end{bmatrix}$

13. Let A and B be $n\times n$ matrices. Multiplying a matrix by a scalar t obeys this formula : $t(AB)=(tA)(tB)$.

14. The product of two $n\times n$ skew-symmetric matrices is skew-symmetric.

15. $\begin{bmatrix}1&3&2\end{bmatrix}\begin{bmatrix}-2\\3\\0\end{bmatrix}=[7]$

16. $\begin{bmatrix}-2\\3\\0\end{bmatrix}\begin{bmatrix}1&3&2\end{bmatrix}=[7]$

17. $\begin{bmatrix}1&3&2\end{bmatrix}\begin{bmatrix}2\\3\\0\end{bmatrix}=\begin{bmatrix}2&6&2\\3&9&6\\0&0&0\end{bmatrix}$

18. The only $n\times n$ matrix that commutes with all other $n\times n$ matrices is I_n.

19. $\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$ Establishes that every 2×2 matrix has an inverse.

20. If A and B are $n\times n$ matrices such that $AB=I$, then $BA=I$ also.

21. If A and B are invertible $n\times n$ matrices, then so is AB, and furthermore, $(AB)^{-1}=A^{-1}B^{-1}$.

22. If A and B are $n\times n$ matrices such that $AB=I$, then $A^T B^T = I$.

23. If A is a square matrix such that $A^k = I$ for some positive integer k, then A is invertible.

24. If A and B are invertible $n\times n$ matrices, then $A+B\neq 0$ and $AB\neq 0$.

25. Let α be a scalar. Let Q be an $n\times n$ matrix. If $Q\alpha = 0$, then $\alpha = 0$.

26. If A has only nonnegative entries and is invertible, then A^{-1} has only nonnegative entries.

27. If A and B are $n\times n$ matrices such that $BA=I$, then B^T is invertible.

28. A 2×2 matrix $\begin{bmatrix} a & c \\ p & q \end{bmatrix}$ is invertible if and only if $aq\neq pc$.

29. Let $A=\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 5 \\ 1 & 1 & 2 \end{bmatrix}$

Then the first row in A^{-1} is $[1, \ -1, \ 0]$.

30. For square matrices, if $AB=0$, then $BA=0$.

31. If the matrix $\begin{bmatrix} 2\beta & 4 \\ 4 & 2\beta \end{bmatrix}$ is noninvertible, then β must be 2.

32. If A is an $n\times n$ matrix containing no zeros, then A is invertible.

33. If A and B are $n\times n$ matrices such that $AB=-I$, then A is invertible.

34. If A and B are $n\times n$ matrices such that AB is invertible, then A is invertible.

35. Let A be an $m\times n$ matrix, and let B be an $n\times m$ matrix. If $n\neq m$ and AB is invertible, then so is BA.

36. If A and B are $n\times n$ invertible matrices, then $AB=BA$.

37. If ABC is invertible, then so is BCA.

38. If A and B are $n\times n$ invertible matrices, then $A\pm B$ is invertible.

39. For invertible $n\times n$ matrices A and B, $B^{-1}=A^{-1}-B^{-1}(B-A)A^{-1}$.

40. If $(A-A^T)A^{-1}\neq 0$, then A is nonsingular skew-symmetric matrix.

Assignment

SCQ

1. If A and B are two odd order skew symmetric matrices such that $AB=BA$, then what is the matrix AB?
   1. An orthogonal matrix
   2. A skew-symmetric matrix
   3. A symmetric matrix
   4. An identity matrix

2. If A and B are symmetric matrices of the same order, then which one of the following is not correct?
   1. $A+B$ is a symmetric matrix.
   2. $AB-BA$ is a symmetric matrix.
   3. $AB+BA$ is a symmetric matrix.
   4. $A+A^T$ and $B+B^T$ are symmetric matrices.

3. Let A and B be any two $n\times n$ matrices and

$\text{tr}(A)=\sum_{i=1}^{n}a_{ii}$ and $\text{tr}(B)=\sum_{i=1}^{n}b_{ii}$.

Consider the following statement

I. $\text{tr}(AB)=\text{tr}(BA)$

II. $\text{tr}(A+B)=\text{tr}(A)+\text{tr}(B)$

Which of the following statement given above is/are correct ?

1. I only
2. II only
3. Both I and II
4. Neither I nor II

4. Let $A=\begin{pmatrix}2 & 0\\ 3 & 5\end{pmatrix}$ be expressed as $P+Q$, where P is symmetric matrix and Q is skew-symmetric matrix. Which one of the following is correct ?

1. $Q=\begin{pmatrix}\frac{1}{2} & -\frac{3}{2}\\ \frac{3}{2} & 0\end{pmatrix}$
2. $Q=\begin{pmatrix}0 & \frac{3}{2}\\ \frac{3}{2} & 0\end{pmatrix}$
3. $Q=\frac{1}{2}\begin{pmatrix}0 & -3\\ 3 & 0\end{pmatrix}$
4. $Q=\frac{1}{2}\begin{pmatrix}2 & 3\\ 0 & 5\end{pmatrix}$

5. What is the determinant of the following matrix ?

$$\begin{bmatrix}1 & 0 & 0 & .. & .. & 0\\ 1 & \frac{1}{2} & 0 & .. & .. & 0\\ 1 & \frac{1}{2} & \frac{1}{3} & .. & .. & 0\\ .. & .. & .. & .. & .. & ..\\ 1 & \frac{1}{2} & \frac{1}{3} & .. & .. & \frac{1}{n}\end{bmatrix}$$

1. n^2+n+1
2. $\frac{1}{n^2}+\frac{1}{n}+1$
3. $\frac{1}{n!}$
4. $\frac{n(n+1)}{2}$

6. If A be a non-zero square matrix of order n, then

1. the matrix $A+A'$ is anti-symmetric, but the matrix is $A-A'$ symmetric.
2. the matrix $A+A'$ is symmetric, but the matrix $A-A'$ is anti-symmetric.
3. Both $A+A'$ and $A-A'$ are symmetric.
4. Both $A+A'$ and $A-A'$ are anti-symmetric.

7. If C is a non-singular matrix and

$$B=C\begin{bmatrix}0 & x & y\\ 0 & 0 & z\\ 0 & 0 & 0\end{bmatrix}C^{-1}$$, then

1. $B^2=1$
2. $B^2=0$
3. $B^3=1$
4. $B^3=0$

8. Suppose, P is an $n\times n$ real matrix such that the k^{th} diagonal element of PP^T is zero. Consider the following statements

I. The kth row of P is zero.

II. The kth row of PP^T is zero.

III. The kth column of P is zero.

IV. The kth column of PP^T is zero.

Select the correct answer using the codes given below

1. I and III
2. I,II and IV
3. II,III and IV
4. I,II and III

(GATE)

9. The determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ is}$$

1. 0 2. −9 3. −27 4. 1

(CSIR NET Dec 2011)

10. If A and B are two $n\times n$ matrices over $\mathbb{R}$ and $\alpha\in\mathbb{R}$, then

1. $\det(\alpha A+B)=\alpha\det(A)+\det(B)$
2. $\det(\alpha A-B)=\alpha\det(A)+\det(B)$
3. $\det(\alpha A\cdot B)=\alpha\det(A)\cdot\det(B)$
4. $\det(\alpha A\cdot B)=\alpha^n\det(A)\cdot\det(B)$

11. Let A be a 2×2 matrix which satisfy $A^2-A=0$, then

1. A is either $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ or $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$
2. there exists infinitely many such matrices
3. there exists no such matrix
4. A must be diagonal

12. The number of different $n\times n$ symmetric matrices with each element being either 0 or 1 where $n=5$

1. $5!$ 2. 5^5
3. 2^{15} 4. 5^{15}

13. Let A be an improper orthogonal matrix, then, adj A is equal to

1. A 2. A^T
3. $-A$ 4. $-A^T$

14. Which is not correct, if $T_r(A)=$ Trace of A, then

1. $T_r(A+B)=T_r(A)+T_r(B)$
2. $T_r(AB)=T_r(BA)$
3. $T_r(A)=T_r(CAC^{-1}), C$ is non-singular
4. For any A there exists B (both are square of same order) s.t. $(AB-BA)=I_n$

15. If B is a non-singular matrix and A is a square matrix. Then, $\det(B^{-1}AB)$ is equal to

1. $\det(BAB)$ 2. $\det(A)$
3. $\det(B^{-1})$ 4. $\det(A^{-1})$

16. Every skew-symmetric matrix of odd order is

1. singular
2. non-singular
3. invertible
4. Skew-Hermitian

17. If $(A+B)^{-1}$ exists, then

1. A^{-1} and B^{-1} both exist
2. A^{-1} and B^{-1} do not exist
3. atleast one of A^{-1} and B^{-1} exists
4. nothing can be said

18. $A, B, (A+B)$ are non-singular matrices. Then, $[B(A+B)^{-1}A]^{-1}$ is equal to

1. $A+B$ 2. $A^{-1}+B^{-1}$
3. $A^{-1}+B^{-1}+I$ 4. AB

19. If A and B are two matrices of the same order such that $AB=BA$, then

1. A is diagonal and B is any matrix
2. A and B are both diagonal
3. A is scalar and B is diagonal matrix
4. None of the above

20. If A and B are Hermitian, then select the incorrect one

1. $AB+BA$ is hermitian
2. $AB-BA$ is skew-hermitian
3. $B^H B$ is hermitian
4. $A+A^H$ is skew-hermitian

21. The columns of an orthogonal matrix forms

1. an orthogonal set of vectors
2. an orthonormal set of vectors
3. a linearly independent set
4. All of the above

22. If $A=\begin{bmatrix}2 & 6\\3 & 9\end{bmatrix}, B=\begin{bmatrix}3 & x\\y & 2\end{bmatrix}$, then in order that $AB=0$. The values of x and y will be, respectively

1. -6 and -1 2. 6 and 1
3. -6 and -3 4. -5 and 4

23. Let $A_{n\times n}=(a_{ij}), n\ge 3$, where $a_{ij}=(b_i^2-b_j^2), i, j=1,2,....,n$ for some distinct real numbers $b_1, b_2, ..., b_n$. Then $\det(A)$ is

1. $\prod_{i<j}(b_i-b_j)$. 2. $\prod_{i<j}(b_i+b_j)$.
3. 0 4. 1

(CSIR NET Dec 2013)

24. Let A be a 5×5 matrix with real entries such that the sum of the entries in each row of A is 1. Then the sum of all the entries in A^3 is

1. 3 2. 15 3. 5 4. 125

(CSIR NET June 2014)

25. Given the permutation

$\sigma=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\3 & 1 & 2 & 5 & 4\end{pmatrix}$, the matrix A is defined to be the one whose i-th column is the $\sigma(i)$-th column of the identity matrix I. Which of the following is correct ?

1. $A=A^3$ 2. $A=A^5$
3. $A=A^4$ 4. $A=A^2$

(CSIR NET June 2014)

26. Let J denote a 101×101 matrix with all the entries equal to 1 and let I denote the identity matrix of order 101. Then the determinant of $J-I$ is

1. 101 2. 1
3. 0 4. 100

(CSIR NET June 2014)

27. For the matrix A as given below, which of them satisfy $A^6=I$?

1. $A=\begin{pmatrix}\cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0\\-\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0\\0 & 0 & 1\end{pmatrix}$

2. $A=\begin{pmatrix}1 & 0 & 0\\0 & \cos\frac{\pi}{3} & \sin\frac{\pi}{3}\\0 & -\sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{pmatrix}$

3. $A=\begin{pmatrix} \cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{6} & 0 & \cos\frac{\pi}{6} \end{pmatrix}$

4. $A=\begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} & 0 \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(CSIR NET June 2014)

28. Let A, B be $n\times n$ matrices such that $BA+B^2=I-BA^2$ where I is the $n\times n$ identity matrix. Which of the following is always true ?

1. A is nonsingular
2. B is nonsingular
3. $A+B$ is nonsingular
4. AB is nonsingular

(CSIR NET Dec 2014)

29. The determinant of the $n\times n$ permutation matrix

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & ⋰ & & \\ 1 & & & \end{bmatrix}_{n\times n}$$

1. $(-1)^n$
2. $(-1)^{\left[\frac{n}{2}\right]}$
3. -1
4. 1

(CSIR NET Dec 2014)

30. The determinant $\begin{vmatrix} 1 & 1+x & 1+x+x^2 \\ 1 & 1+y & 1+y+y^2 \\ 1 & 1+z & 1+z+z^2 \end{vmatrix}$ is equal to

1. $(z-y)(z-x)(y-x)$
2. $(x-y)(x-z)(y-z)$
3. $(x-y)^2(y-z)^2(z-x)^2$
4. $(x^2-y^2)(y^2-z^2)(z^2-x^2)$

(CSIR NET Dec 2014)

31. Let $S=\{A : A=[a_{ij}]_{5\times 5}$, $a_{ij}=0$ or $1\ \forall i,j$, $\sum_j a_{ij}=1\ \forall i$ and $\sum_i a_{ij}=1\ \forall j\}$ Then the number of elements in S is

1. 5^2
2. 5^5
3. $5!$
4. 55

(CSIR NET June 2011)

32. Let $\alpha=e^{\frac{2\pi i}{5}}$ and the matrix

$$M=\begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}.$$

Then, the trace of the matrix $I+M+M^2$ is

1. -5
2. 0
3. 3
4. 5

(GATE 2012)

33. If $A=\begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$, then the trace of A^{102} is

1. 0
2. 1
3. 2
4. 3

(GATE 2009)

34. For which value of x will the matrix given below become singular ?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

1. 4 2. 6

3. 8 4. 12

35. If $A=\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, then A is

1. idempotent 2. nilpotent

3. involutory 4. periodic

36. Let $f(x)=x^2-5x+6$, and

$A=\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then $f(A)$ is equal to

1. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & -1 & -5 \\ -1 & -1 & 4 \\ -3 & -10 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -1 & 4 \\ -1 & 4 & -10 \\ 4 & -3 & -5 \end{bmatrix}$ 4. None of these

37. Multiplication of matrices E and F is G. Matrices E and G are as follows :

$E=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $G=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then, the value of matrix F is

1. $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

38. Let A be a 5×5 matrix such that sum of elements of each row is 2 then the sum of all elements of A^3 is

1. 45 2. 40

3. 90 4. None of these

39. Let A and B are two 3×3 complex matrices such that sum of elements of each row of A is $1+i$ and that if B is i then sum of all nine elements of AB is

1. $3+3i$ 2. $3-3i$

3. $-3+3i$ 4. $-3-3i$

40. Let P be a $n\times n$ matix with integral entries and $Q=P+\frac{1}{2}I$, where I denotes the $n\times n$ identity matrix. Then, Q is

1. idempotent, i.e., $Q^2=Q$

2. invertible

3. nilpotent

4. unipotent, i.e., $Q-I$ is nilpotent

(GATE 2004)

41. Let $D_1 = \det\begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}$ and

$D_2 = \det\begin{pmatrix} -x & a & -p \\ y & -b & q \\ z & -c & r \end{pmatrix}$. Then

1. $D_1 = D_2$
2. $D_1 = 2D_2$
3. $D_1 = -D_2$
4. $2D_1 = D_2$

(CSIR NET Dec 2016)

42. Consider the matrix $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$,

where $\theta = \frac{2\pi}{31}$. Then A^{2015} equals

1. A
2. I
3. $\begin{pmatrix} \cos 13\theta & \sin 13\theta \\ -\sin 13\theta & \cos 13\theta \end{pmatrix}$
4. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(CSIR NET Dec 2016)

43. Let A and B be real invertible matrices such that $AB = -BA$. Then

1. $\text{Trace}(A) = \text{Trace}(B) = 0$
2. $\text{Trace}(A) = \text{Trace}(B) = 1$
3. $\text{Trace}(A) = 0, \text{Trace}(B) = 1$
4. $\text{Trace}(A) = 1, \text{Trace}(B) = 0$

(CSIR NET June 2017)

44. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Then the smallest positive integer n such that $A^n = I$ is

1. 1
2. 2
3. 4
4. 6

(CSIR NET Dec 2017)

45. The trace of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20}$ is

1. 7^{20}
2. $2^{20} + 3^{20}$
3. $2 \cdot 2^{20} + 3^{20}$
4. $2^{20} + 3^{20} + 1$

(CSIR NET June 2018)

46. For $t \in \mathbb{R}$, define $M(t) = \begin{pmatrix} 1 & t & 0 \\ 1 & 1 & t^2 \\ 0 & 1 & 1 \end{pmatrix}$.

Then which of the following statements is true ?

1. $\det M(t)$ is a polynomial function of degree 3 in t.
2. $\det M(t) = 0$ for all $t \in \mathbb{R}$
3. $\det M(t)$ is zero for infinitely many $t \in \mathbb{R}$
4. $\det M(t)$ is zero for exactly two $t \in \mathbb{R}$

(CSIR NET Dec 2019)

47. Let A and B be 2×2 matrices. Then which of the following is true ?

1. $\det(A+B) + \det(A-B) = \det A + \det B$
2. $\det(A+B) + \det(A-B) = 2\det A - 2\det B$
3. $\det(A+B) + \det(A-B) = 2\det A + 2\det B$
4. $\det(A+B) - \det(A-B) = 2\det A - 2\det B$

(CSIR NET Nov 2020)

MCQ

1. Let $A=(a_{ij})$ be an $n\times n$ complex matrix and let A^* denote the conjugate transpose of A. Which of the following statements are necessarily true?
   1. If A is invertible, then $tr(A^*A)\neq 0$, *i.e*, the trace of A^*A is non zero.
   2. If $tr(A^*A)\neq 0$, then A is invertible.
   3. If $|tr(A^*A)|<n^2$, then $|a_{ij}|<1$ for some i,j.
   4. If $tr(A^*A)=0$, then A is zero matrix.

(CSIR NET Dec 2012)

2. Let A be a 5×5 skew–symmetric matrix with entries in $\mathbb{R}$ and B be the 5×5 symmetric matrix whose $(i,j)^{th}$ entry is the binomial coefficient $\binom{i}{j}$ for $1\le i\le j\le 5$. Consider the 10×10 matrix, given in block form by $C=\begin{pmatrix} A & A+B \\ 0 & B\end{pmatrix}$. Then
   1. $\det C=1$ or -1
   2. $\det C=0$
   3. trace of C is 0
   4. trace of C is 5

(CSIR NET Dec 2012)

3. Let S denote the set of all primes p such that the following matrix is invertible when considered as a matrix with entries in $\mathbb{Z}/p\mathbb{Z}$, $A=\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ -2 & 0 & 2\end{pmatrix}$. Which of the following statements are true ?
   1. S contains all the prime numbers.
   2. S contains all the prime numbers greater than 10.
   3. S contains all the prime numbers other than 2 and 5.
   4. S contains all the odd prime numbers.

(CSIR NET June 2013)

4. The matrix $A=\begin{pmatrix} 5 & 9 & 8 \\ 1 & 8 & 2 \\ 9 & 1 & 0\end{pmatrix}$ satisfies
   1. A is invertible and the inverse has all integer entries.
   2. det(A) is odd.
   3. det(A) is divisible by 13.
   4. det(A) has atleast two prime divisors.

(CSIR NET Dec 2014)

5. Which of the following(s) is/are correct ?
   1. The transpose of a symmetric matrix need not be symmetric matrix.
   2. If A and B are symmetric matrix of same order, then $AB+BA$ must be symmetric matrix.
   3. If A is symmetric matrix, then all positive integral powers of A are symmetric matrices.
   4. If A is any square matrix, then $A+A'$ is always symmetric matrix.

6. Which of the following(s) is/are correct ?

1. If A is orthogonal matrix, then A is non-singular and $A^{-1}=A'$
2. If A is orthogonal matrix, then $|A|=\pm 1$
3. Transpose of an orthogonal matrix is orthogonal
4. None of the above

7. Which of the following(s) is/are correct ?
1. If A is unitary matrix, then A' is also unitary matrix.
2. Inverse of unitary matrix is unitary matrix.
3. $A=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary matrix.
4. None of the above

8. Which of the following(s) is/are correct ?
1. In a skew hermitian matrix, the element on the principal diagonal must be purely imaginary.
2. If A is hermitian matrix, then iA is skew hermitian matrix.
3. If A is any square matrix, then $A-A^*$ is skew hermitian matrix.
4. All of the above.

9. Let $A=[a_{ij}]$ be an $n\times n$ matrix such that a_{ij} is an integer for all i,j. Let $AB=I$ with $B=[b_{ij}]$ (where I is the identity matrix). For a square matrix C, det C denotes its determinant. Which of the following statements is true ?
1. If $\det A=1$ then $\det B=1$
2. A sufficient condition for each b_{ij} to be an integer is that $\det A$ is an integer.
3. B is always an integer matrix.
4. A necessary condition for each b_{ij} to be an integer is $\det A\in\{-1,+1\}$

(CSIR NET Dec 2016)

10. Let m,n,r be natural numbers. Let A be an $m\times n$ matrix with real entries such that $(AA^t)^r=I$, where I is the $m\times m$ identity matrix and A^t is the transpose of the matrix A. We can conclude that
1. $m=n$
2. AA^t is invertible
3. A^tA is invertible
4. if $m=n$, then A is invertible

(CSIR NET June 2017)

11. Let $A=((a_{ij}))$ be a 3×3 complex matrix identity the correct statements
1. $\det\left(\left((-1)^{i+j}a_{ij}\right)\right)=\det A$
2. $\det\left(\left((-1)^{i+j}a_{ij}\right)\right)=-\det A$
3. $\det\left(\left(\left(\sqrt{-1}\right)^{i+j}a_{ij}\right)\right)=\det A$
4. $\det\left(\left(\left(\sqrt{-1}\right)^{i+j}a_{ij}\right)\right)=-\det A$

(CSIR NET June 2019)

12. Let $n\geq 1$ and $\alpha,\beta\in\mathbb{R}$ with $\alpha\neq\beta$. Suppose $A_n(\alpha,\beta)=[a_{ij}]$ is an $n\times n$ matrix such that $a_{ii}=\alpha$ and $a_{ij}=\beta$ for $i\neq j$, $1\leq i,j\leq n$. Let D_n be the determinant of $A_n(\alpha,\beta)$. Which of the following statements are true ?
1. $D_n=(\alpha-\beta)D_{n-1}+\beta$ for $n\geq 2$

2. $\dfrac{D_n}{(\alpha-\beta)^{n-1}} = \dfrac{D_{n-1}}{(\alpha-\beta)^{n-2}} + \beta$ for $n \geq 2$

3. $D_n = (\alpha+(n-1)\beta)^{n-1}(\alpha-\beta)$ for $n \geq 2$

4. $D_n = (\alpha+(n-1)\beta)(\alpha-\beta)^{n-1}$ for $n \geq 2$

(CSIR NET Dec 2019)

True false key

1. T	2. F	3. T	4. T
5. F	6. T	7. F	8. F
9. F	10. F	11. F	12. F
13. F	14. F	15. T	16. F
17. F	18. F	19. F	20. T
21. F	22. T	23. T	24. F
25. F	26. F	27. T	28. T
29. T	30. F	31. F	32. F
33. T	34. T	35. F	36. F
37. F	38. F	39. T	40. F

Assignment key

S C Q

1. 3	2. 2	3. 3	4. 3
5. 3	6. 2	7. 4	8. 2
9. 3	10. 4	11. 2	12. 3
13. 4	14. 4	15. 2	16. 1
17. 4	18. 2	19. 4	20. 4
21. 4	22. 1	23. 3	24. 3
25. 3	26. 4	27. 2	28. 2
29. 2	30. 1	31. 3	32. 4
33. 4	34. 1	35. 3	36. 1
37. 3	38. 2	39. 3	40. 2
41. 3	42. 2	43. 1	44. 4
45. 3	46. 4	47. 3	

M C Q

1. 1,3,4	2. 2,4	3. 2,3	4. 3,4
5. 2,3,4	6. 1,2,3	7. 1,2,3	8. 2,3
9. 1,4	10. 2,4	11. 1,3	12. 2,4